E-LECTURE SERIES PPPJJ,USM

JIM 417
PARTIAL DIFFERENTIAL EQUATIONS

FOURIER SERIES

Fourier Series of Even and Odd Functions

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Learning Outcome(s)

By the end of this e-lecture, students should be able to:

- Classify whether the function is an even or odd function.
- Utilize and compute the Fourier Series of Even and Odd Functions.

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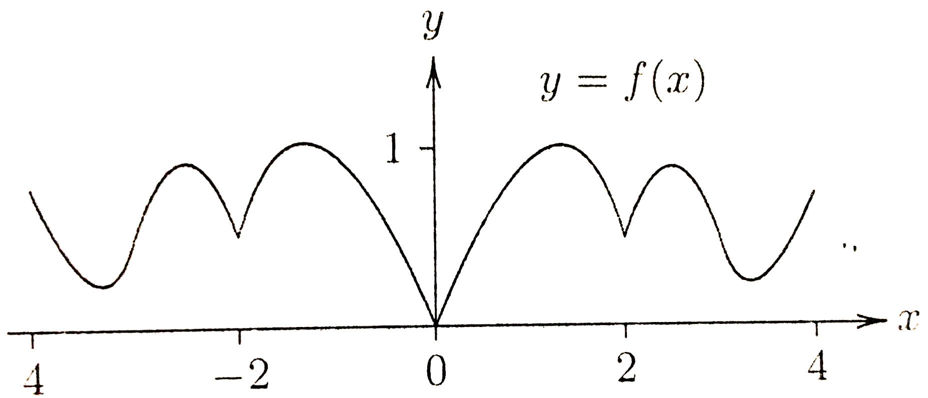
Even and Odd Functions

Even function is any function f such that

$$f(-x) = f(x), \text{ for all } x \text{ in its domain.}$$

Example: $\cos(x), sec(x),$ any constant function
$x^2, x^4, x^6, x^8, \ldots, x^{-2}, x^{-4}, \ldots$

The shape of the graph: Graph is symmetrical about the y-axis.



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Even and Odd Functions

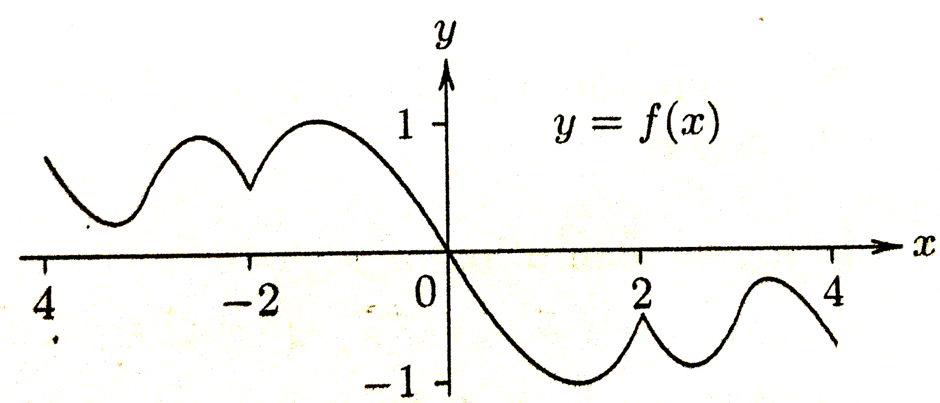
Odd function is any function f such that

$$f(-x) = -f(x), \text{ for all } x \text{ in its domain.}$$

Example: $\sin(x), \tan(x), \csc(x), \cot(x)$ any constant function

$x \quad, x^3, x^5, x^7, \dots, x^{-1}, x^{-3}, \dots$

The shape of the graph: Graph is symmetrical about the origin.



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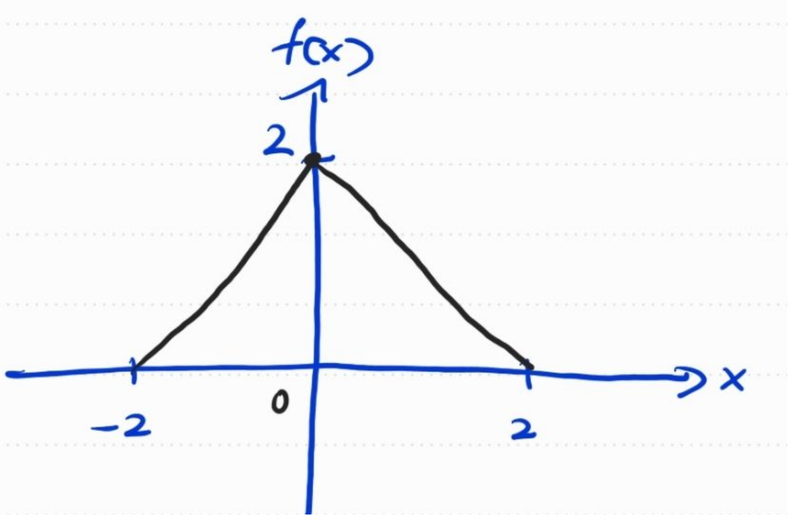
Example 1

Sketch the graph of the following function. State whether the function is even, odd or neither.

$$f(x) = \begin{cases} 2 + x, & -2 < x < 0, \\ 2 - x, & 0 < x < 2; \end{cases}$$

Solution:

The graph is symmetrical about y-axis, so it is an even function.



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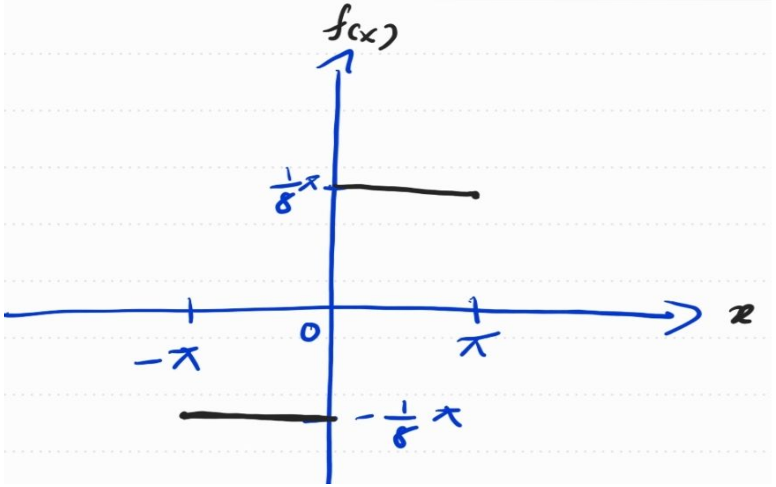
Example 2

Sketch the graph of the following function. State whether the function is even, odd or neither.

$$f(x) = \begin{cases} -\dfrac{1}{8}\pi, & -\pi < x < 0, \\ \dfrac{1}{8}\pi, & 0 < x < \pi; \end{cases}$$

Solution:

The graph is symmetrical about origin, so it is an odd function.



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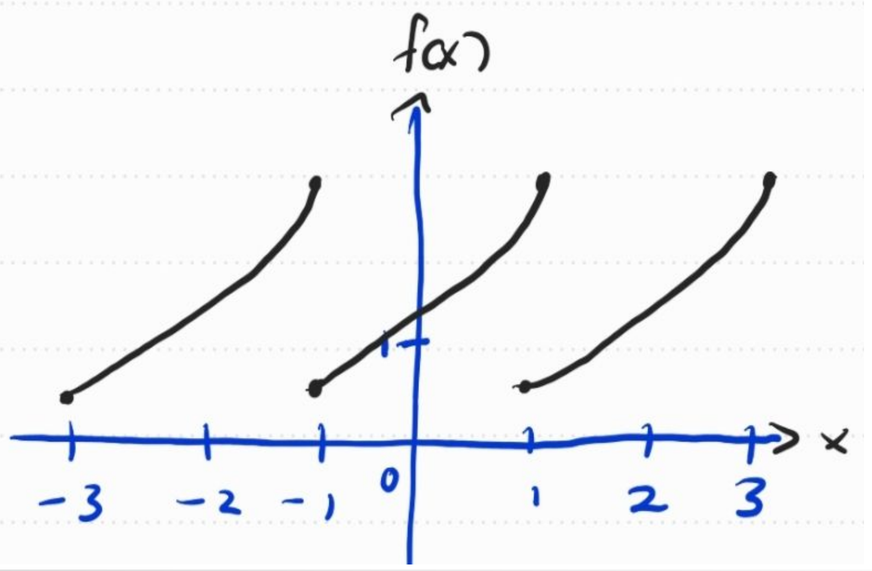
Example 3

Sketch the graph of the following function. State whether the function is even, odd or neither.

$$f(x) = e^x, \qquad -1 < x < 1$$
$$f(x) = f(x+2)$$

Solution:

The graph is not symmetrical at origin or y-axis, so it is neither even nor odd function.



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Theorem 5.4 (Properties of products of Functions)

- If $f(x)$ and $g(x)$ are both **even** functions, then $f(x)g(x)$ is **even**.
- If $f(x)$ and $g(x)$ are both **odd** functions, then $f(x)g(x)$ is **even**.
- If $f(x)$ is an **even** function and $g(x)$ is an **odd** function, then $f(x)g(x)$ is **odd** .

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Calculus Properties of Even and Odd Functions

Suppose f is an even function, continuous on $-L \leq x \leq L$, then

$$\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx.$$

Suppose f is an odd function, continuous on $-L \leq x \leq L$, then

$$\int_{-L}^{L} f(x)\,dx = 0.$$

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Example 4

Evaluate the integrals using properties of even/odd functions.

$$\int_{-\pi}^{\pi} x^2 \sin \pi x \, dx$$

$$\int_{-\pi}^{\pi} x^2 \sin \pi x \, dx$$

x^2 is an even function

$\sin \pi x$ is an odd function

$x^2 \sin \pi x$ is an odd function.

$\therefore, \int_{-L}^{L} f(x)\,dx = 0$

$$\int_{-\pi}^{\pi} x^2 \sin \pi x \, dx = 0 \; \#$$

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Example 5

Evaluate the integrals using properties of even and odd functions.

$$\int_{-\pi}^{\pi} x \sin x \, dx$$

$$\int_{-\pi}^{\pi} x \sin x \, dx$$

- x is an odd function
- $\sin x$ is an odd function
- $x \sin x$ is an even function

$$\int$$

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Fourier Series of Even and Odd Functions

- A Fourier series representation is a combination of cosine and sine term.
- It is well-known that ***cosine functions*** are **even**, whereas ***sine functions*** are **odd**.
- If a function is symmetric, either even or odd, this property will **determine which terms are absent from Fourier series expansion** of the function.
- Thus, such knowledge will **greatly reduce** the loads of determining the coefficient of the expansion.

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Fourier Series of Even and Odd Functions

- In some of the problems that we encounter, the Fourier coefficients a_0, a_n or b_n become **zero** after integration.

- Finding zero coefficients in such problems is time consuming and can be avoided. With knowledge of **even and odd functions**, a zero coefficient may be predicted without performing the integration.

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Fourier Series of Even Function

Suppose $f(x)$ be an **even function** defined in the interval $-L < x < L$ and $f(x)$ be periodic function with period of $\mathrm{T} = 2L$. The Fourier series of $f(x)$ is given by:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right),$$

where

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx = \frac{2}{L}\int_{0}^{L} f(x)dx\,,$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \qquad (n=1,2,3,\ldots),$$

$$b_n = 0$$

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Fourier Series of Even and Odd Functions

Fourier Series of Odd Function

Suppose $f(x)$ be an **odd function** defined in the interval $-L < x < L$ and $f(x)$ be periodic function with period of $T = 2L$. The Fourier series of $f(x)$ is given by:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

where

$$a_0 = 0, \qquad a_n = 0,$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \qquad n=1,2,3,\ldots$$

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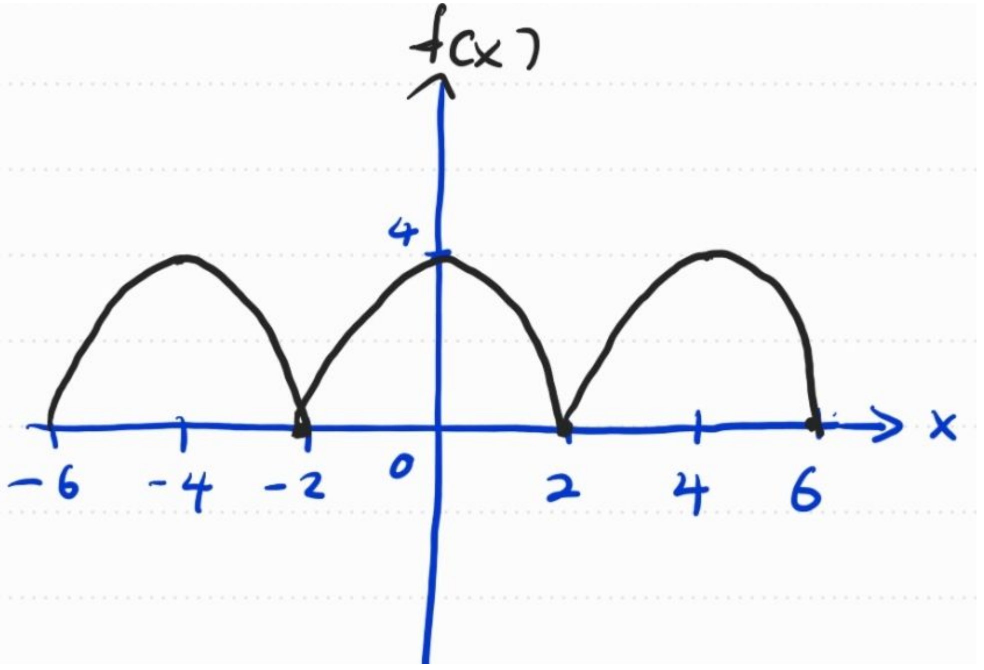
Question 6

A periodic function $f(x)$ is defined by

$$f(x) = 4 - x^2, -2 < x < 2$$

$$f(x) = f(x+4).$$

(a) Sketch the graph of the function over $-6 \leq x \leq 6$.



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Question 6

A periodic function $f(x)$ is defined by

$$f(x) = 4 - x^2, -2 < x < 2$$

$$f(x) = f(x+4).$$

(b) Compute the Fourier series of $f(x)$.

Solution:

From the graph in (a), the function is an even function.

$$a_0 = \frac{2}{2}\int_0^2 (4 - x^2)dx$$

$$= \left[4x - \frac{x^3}{3}\right]_0^2 = \frac{16}{3}$$

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Question 6

Solution:

$$a_n = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \int_0^2 (4 - x^2)\cos\frac{n\pi x}{2}\,dx$$

$$= \int_0^2 (4)\cos\frac{n\pi x}{2}\,dx - \int_0^2 (x^2)\cos\frac{n\pi x}{2}\,dx$$

$$= \left[\frac{8}{n\pi}\sin\frac{n\pi x}{2}\right]_0^2 - \left[\frac{8x}{n^2\pi^2}\cos\frac{n\pi x}{2} + \left(\frac{2x^2}{n\pi} - \frac{16}{n^3\pi^3}\right)\sin\frac{n\pi x}{2}\right]_0^2$$

$$= -\frac{16}{n^2\pi^2}\cos n\pi$$

$$= \frac{16}{n^2\pi^2}(-1)(-1)^n$$

$$= \frac{16}{n^2\pi^2}(-1)^{n+1}$$

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Question 6

Solution:

Hence, the Fourier series corresponding to the function $f(x)$ is

$$f(x) = \frac{(\frac{16}{3})}{2} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos\left(\frac{n\pi x}{2}\right)$$

Therefore,

$$f(x) = \frac{8}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos\left(\frac{n\pi x}{2}\right)$$

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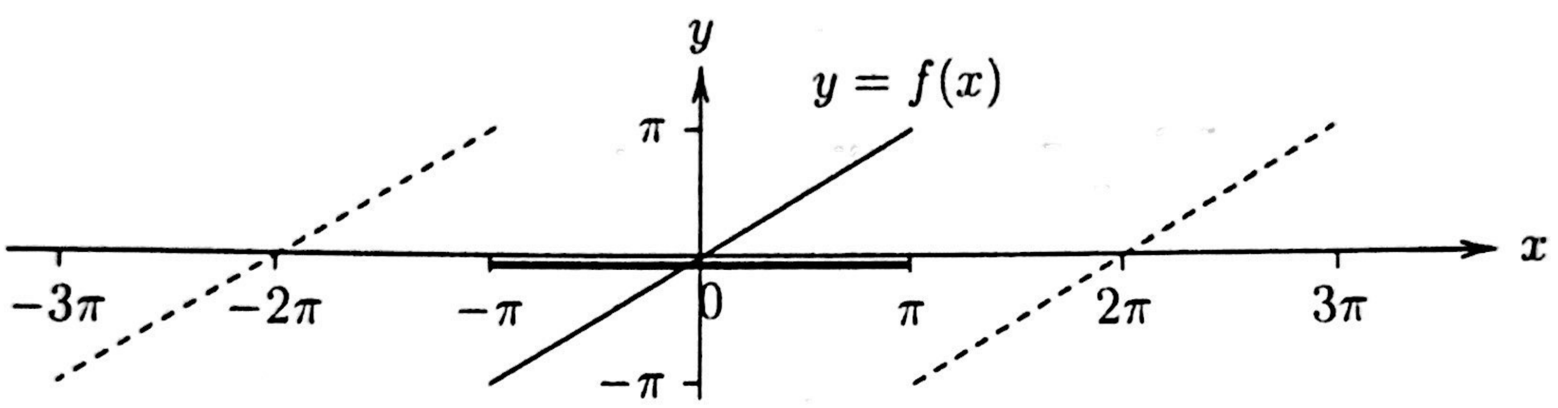
Question 7

A periodic function $f(x)$ is defined by

$$f(x) = x\,, -\pi < x < \pi$$

$$f(x) = f(x + 2\pi).$$

(a) Sketch the graph of the function over $-3\pi \leq x \leq 3\pi$.



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Question 7

(b) Compute the Fourier series of $f(x)$.

Solution

From the graph in (a), the function is an odd function.

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin\frac{n\pi x}{\pi}\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin\frac{n\pi x}{\pi}\,dx$$

$$= \frac{2}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(-\frac{\pi\cos n\pi}{n} + \frac{\sin n\pi}{n^2}\right) - \left(0 + \frac{\sin 0}{n^2}\right)\right]$$

$$= \frac{2}{\pi}\left(-\frac{\pi\cos n\pi}{n}\right) = -\frac{2}{n}\cos n\pi$$

$$= \frac{2}{n}(-1)(-1)^n$$

$$= \frac{2}{n}(-1)^{n+1}$$

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Question 7

(b) Compute the Fourier series of $f(x)$.

Solution

$$b_n = \frac{2}{n}(-1)^{n+1}$$

The first few values of b_n are

$$b_1 = \frac{2}{1}, b_2 = -\frac{2}{2}, b_3 = \frac{2}{3}, b_4 = -\frac{2}{4}, b_5 = \frac{2}{5}, b_6 = -\frac{2}{6}, \dots \dots$$

Hence, the Fourier series:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$= 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \frac{\sin n\pi x}{\pi}$$

$$= 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$

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HALF-RANGE FOURIER SERIES

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