JIM 417: Partial Differential Equations

WEBEX CLASS 2

Chapter 1: Laplace Transform (Part B)

19th October 2019

5.00-6.00 pm (Saturday)

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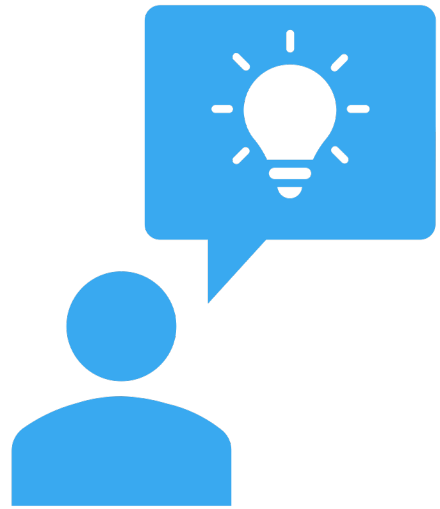
Learning Outcome(s)

By the end of this WEBEX, students should be able to:

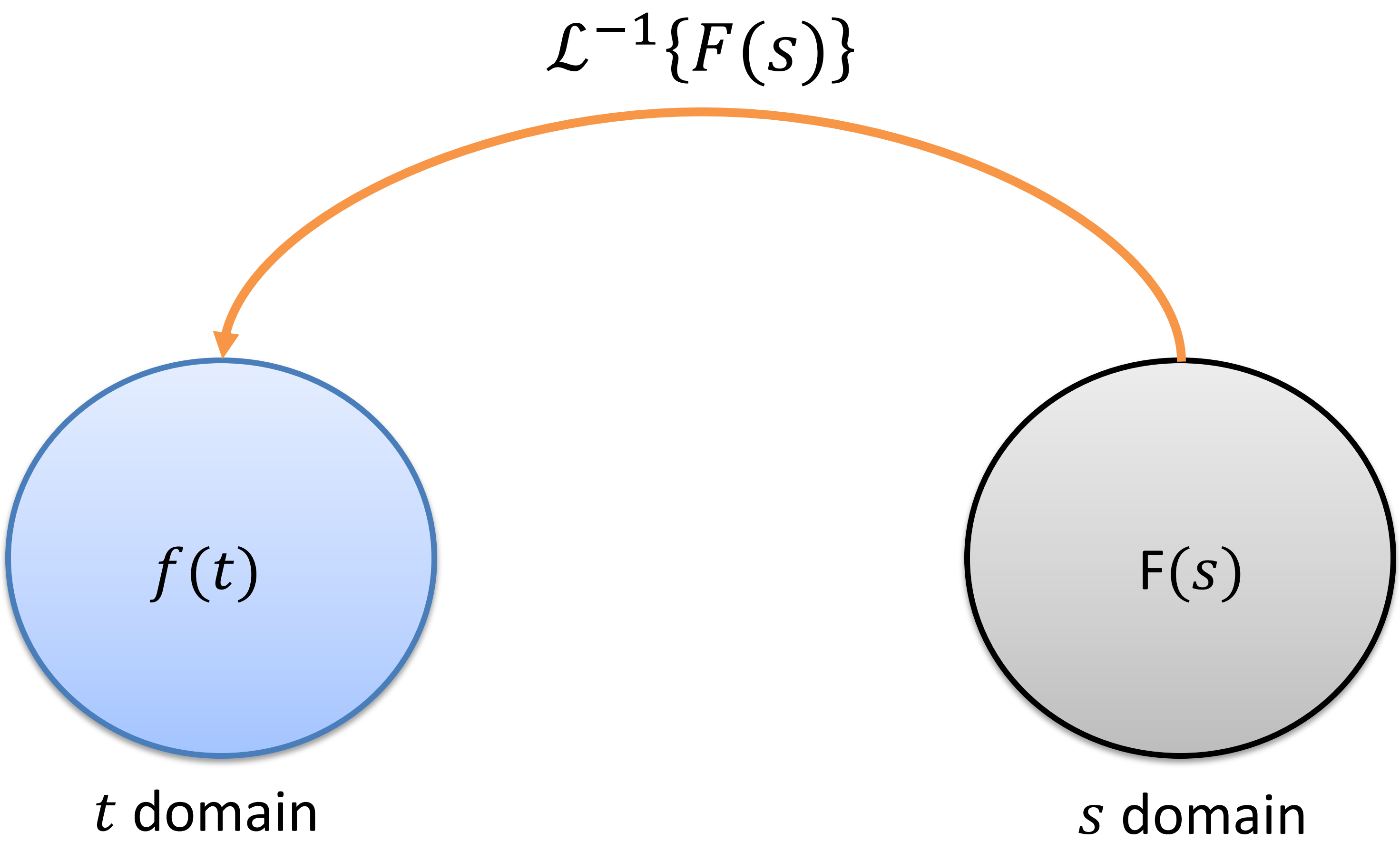
- Define the inverse Laplace transforms.
- Determine the inverse Laplace transforms by using related formulae.
- Determine the inverse Laplace transforms by using partial fractions.
- Determine the inverse Laplace transforms by using convolution theorem.
- Use the Laplace transform to solve initial value problems.

1.4 Inverse Laplace Transform



By definition, the inverse Laplace transform operator, $\mathcal{L}^{-1}$, converts an s-domain function back to the corresponding time domain function:

1.4 Inverse Laplace Transform



1.4 Inverse Laplace Transform

Definition 1.8 (Inverse Laplace Transform)

If $\mathcal{L}\{f(t)\} = F(s)$, then $f(t)$ is called inverse Laplace transform of $F(s)$ and is written as

$$\mathcal{L}^{-1}\{F(s)\} = f(t).$$

This transform is written symbolically as $\mathcal{L}^{-1}\{F(s)\}$, where $\mathcal{L}^{-1}$ is interpreted as inverse Laplace operator.

COMMON MISCONCEPTION:

$$\mathcal{L}^{-1} = \frac{1}{\mathcal{L}}$$

THE CORRECT CONCEPT:

$$\mathcal{L}^{-1} \neq \frac{1}{\mathcal{L}}$$

Table of the Laplace Transform of Elementary Functions:

$$\mathcal{L}^{-1}\{F(s)\} = f(t).$$

$F(s)$	$f(t)$
$\frac{a}{s}$	a
$\frac{n!}{s^{n+1}}$	$t^n, \quad n = 1, 2, 3, \ldots$
$\frac{1}{s-a}$	e^{at}
$\frac{a}{s^2+a^2}$	$\sin at$
$\frac{s}{s^2+a^2}$	$\cos at$
$\frac{a}{s^2-a^2}$	$\sinh at$
$\frac{s}{s^2-a^2}$	$\cosh at$

Example 1

Find the inverse Laplace transforms of the following expressions.

$$\frac{7}{s}$$

Solution:

$$\mathcal{L}^{-1}\left\{\frac{7}{s}\right\}$$

$$\mathcal{L}^{-1}$$

Example 2

Find the inverse Laplace transforms of the following expressions.

$$\frac{6}{s^4}$$

Solution:

$$= \mathcal{L}^{-1}\left\{\frac{6}{s^4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{3!}{s^{3+1}}\right\}$$

$$= t^3$$

Example 3

Find the inverse Laplace transforms of the following expressions.

$$\frac{1}{s+5}$$

Solution:

$$\mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\}$$

$$\mathcal{L}$$

Example 4

Find the inverse Laplace transforms of the following expressions.

$$\frac{7s}{s^2 - 4}$$

Solution:

$$\mathcal{L}^{-1}\left\{\frac{7s}{s^2-4}\right\}$$

$=$

Example 5

Find the inverse Laplace transforms of the following expressions.

$$\frac{2s}{16s^2 + 9}$$

Solution:

$$\mathcal{L}^{-1}\left\{\frac{2s}{16s^2+9}\right\}$$

$\mathcal{L}^{-}$

1.5 Properties of Inverse Laplace Transform

Theorem 1.11 (Linear Property of Inverse Laplace Transform)

If $\mathcal{L}^{-1}\{ F(s)\} = f(t)$ and $\mathcal{L}^{-1}\{ G(s)\} = g(t)$, and α and β are constants then:

$$\mathcal{L}^{-1}\{\alpha\, F(s) + \beta\, G(s)\} = \alpha\mathcal{L}^{-1}\{F(s)\} + \beta\mathcal{L}^{-1}\{G(s)\}$$
$$= \alpha\, f(t) + \beta\, g(t)$$

Example 6

Find the inverse Laplace transforms of the following expressions.

$$\frac{4s+15}{s^2+9}$$

Solution:

$$= \mathcal{L}^{-1}\left\{\frac{4s+15}{s^2+9}\right\}$$

$$= 4\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + 5\left\{\frac{3}{s^2+9}\right\}$$

$$= 4\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} + 5\left\{\frac{3}{s^2+3^2}\right\}$$

By looking Table of Inverse Laplace:

$$= 4\cos 3t + 5\sin 3t$$

1.5 Properties of Inverse Laplace Transform

Theorem 1.12 (First Shift Property of Inverse Laplace Transform)

If $\mathcal{L}^{-1}\{F(s)\} = f(t)$ and a is a constants then:

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t),$$

or alternatively:

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}\mathcal{L}^{-1}\{F(s)\}.$$

Example 7

Find the inverse Laplace transforms of the following expressions.

$$\frac{3}{2s-6}$$

Solution:

$$= \mathcal{L}^{-1}\left\{\frac{3}{2s-6}\right\}$$

$$= \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

By looking Table of Inverse Laplace:

$$= \frac{3}{2}e^{3t}$$

Example 8

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Find the inverse Laplace transforms of the following expressions.

$$\frac{2s+5}{(s+3)^2}$$

Solution:

$$\frac{2s+5}{(s+3)^2}$$

2s |

Theorem 1.13 **(Second Shift Property of Inverse Laplace Transform)**

If $\mathcal{L}^{-1}\{\,F(s)\} = f(t)$ and a is a constants then:

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = \ f(t-a)\,H(t-a),$$

where

$H(t)$ is a unit step function.

Example 9

Find the inverse Laplace transforms of the following expressions.

$$\frac{e^{-2s}}{s^2}$$

Solution:

From $e^{-as}F(s) = \frac{e^{-2s}}{s^2}$ we have $a = 2$ and $F(s) = \frac{1}{s^2}$

Thus, $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t \qquad and \quad f(t-2)$

By using second shift theory,

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\} = f(t-2)H(t-2) = (t-2)H(t-2)$$

1.6 Inverse Transform of Rational Function

- The inverse Laplace transform of rational functions can be simplified by using partial fractions.
- The rational function of s is considered as:

$$\frac{N(s)}{D(s)}$$

where the degree of $D(s) >$ degree of $N(s)$.

- The rules of partial fractions are vital in finding the inverse Laplace transform in term of fraction or rational function.

1.6 Inverse Transform of Rational Function

Case	Fraction $\frac{N(x)}{D(x)}$	Form of denominator, D(x)	Partial Fraction Form (where A, B and C are unknown constants)
1	$\frac{N(x)}{(ax+b)(cx+d)}$	Linear Factors	$\frac{A}{ax+b} + \frac{B}{cx+d}$
2	$\frac{N(x)}{(ax+b)^2}$	Repeated Linear Factors	$\frac{A}{ax+b} + \frac{B}{(ax+b)^2}$
	$\frac{N(x)}{(ax+b)(cx+d)^2}$	Linear and Repeated Linear Factors	$\frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{(cx+d)^2}$
3	$\frac{N(x)}{(ax+b)(x^2+c^2)}$	Linear and Quadratic (which cannot be factorised) Factors	$\frac{A}{ax+b} + \frac{Bx+C}{x^2+c^2}$

Example 10

Find the inverse Laplace transforms of the following expressions by using partial fractions.

$$\frac{1}{(s+1)(s-2)}$$

SOLUTION:

Express the fraction as sum of two fractions,

$$\frac{1}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2}$$

Equate the denominators,

$$\frac{1}{(s+1)(s-2)} = \frac{A(s-2) + B(s+1)}{(s+1)(s-2)}$$

If $s = 2$

$$A(2-2) + B(2+1) = 1$$

$$3B = 1$$

$$B = \frac{1}{3}$$

Example 10

SOLUTION (CONT..):

If $s = -1$

$$A(-1-2) + B(-1+1) = 1$$
$$-3A = 1$$
$$A = -\frac{1}{3}$$

Therefore,

$$\frac{1}{(s+1)(s-2)} = \frac{(-\frac{1}{3})}{s+1} + \frac{(\frac{1}{3})}{s-2}$$
$$\frac{1}{(s+1)(s-2)} = -\frac{1}{3}\left(\frac{1}{s+1}\right) + \frac{1}{3}\left(\frac{1}{s-2}\right)$$
$$\frac{1}{(s+1)(s-2)} = \frac{1}{3}\left(\frac{1}{s-2}\right) - \frac{1}{3}\left(\frac{1}{s+1}\right)$$

Hence,

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s-2)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{3}\left(\frac{1}{s-2}\right)\right\} - \mathcal{L}^{-1}\left\{\frac{1}{3}\left(\frac{1}{s+1}\right)\right\}$$

Example 10

SOLUTION (CONT..):

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s-2)}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\left(\frac{1}{s-2}\right)\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\left(\frac{1}{s+1}\right)\right\}$$

$$= \frac{1}{3}e^{2t} - \frac{1}{3}e^{-t}$$

$$= \frac{1}{3}\left(e^{2t} - e^{-t}\right)$$

1.7 Convolution Theorem of Laplace Transform

- Sometimes it is possible to write a Laplace transform $H(s)$ as $H(s) = F(s)G(s)$, where $F(s)$ and $G(s)$ are the transforms of known functions f and g, respectively. In this case we might expect $H(s)$ to be the transform of the product of f and g. That is, does

$$H(s) = F(s)G(s) = L\{f\}L\{g\} = L\{f\,g\}?$$

Theorem 1.14 (Convolution Theorem)

If $\mathcal{L}^{-1}\{F(s)\} = f(t)$ and $\mathcal{L}^{-1}\{G(s)\} = g(t)$, then

$$\mathcal{L}^{-1}\{F(s)\,G(s)\} = \int_0^t f(u)\,g(t-u)\,du.$$

Example 11

By using convolutional theorem, find the inverse laplace transforms of the following functions.

$$\frac{1}{(s+1)(s-2)}$$

Solution

Rewrite as:

$$\frac{1}{s+1} \cdot \frac{1}{s-2}$$

Then,

$$F(s) = \frac{1}{s+1}$$

$$G(s) = \frac{1}{s-1}$$

Example 11

By using convolutional theorem, find the inverse laplace transforms of the following functions.

$$\frac{1}{(s+1)(s-2)}$$

Solution

The inverse Laplace transform of $F(s)$

$$f(t) = e^{-t}$$
$$f(u) = e^{-u}$$

The inverse Laplace transform of $\mathrm{G}(s)$

$$\mathrm{g}(t) = e^{2t}$$
$$\mathrm{g}(t-u) = e^{2(t-u)}$$

Example 11

By using convolutional theorem, find the inverse laplace transforms of the following functions.

$$\frac{1}{(s+1)(s-2)}$$

Solution

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s-2)}\right\} = \int_0^t e^{-u} \,.\, e^{2(t-u)} du$$

$$= e^{2t} \int_0^t e^{-3u}\ du$$

$$= e^{2t}\left[-\frac{1}{3}e^{-3u}\right]_0^t$$

$$= \frac{1}{3}\left(e^{2t} - e^{-t}\right)$$

Example 12

By using convolutional theorem, find the inverse laplace transforms of the following functions.

$$\frac{s}{(s^2+1)^2}$$

Solution

Rewrite as:

$$\frac{1}{s^2+1} \cdot \frac{s}{s^2+1}$$

Then,

$F(s) = \frac{1}{s^2+1}$ $\quad G(s) = \frac{s}{s^2+1}$

Example 12

By using convolutional theorem, find the inverse laplace transforms of the following functions.

$$\frac{s}{(s^2+1)^2}$$

Solution

The inverse Laplace transform of $F(s)$

$$f(t) = sin\ t$$
$$f(u) = sin\ u$$

The inverse Laplace transform of G(s)

$$\mathrm{g}(t) = cos\ t$$
$$\mathrm{g}(t-u) = cos\ (t-u)$$

Example 12

By using convolutional theorem, find the inverse laplace transforms of the following functions.

$$\frac{s}{(s^2+1)^2}$$

Solution

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \int_0^t \sin u \cos(t-u)\ du$$

$$= \frac{1}{2}\int_0^t \{\sin u + \cos(t-u)\}\ du$$

$$= \frac{1}{2}\left[u \sin t - \frac{1}{2}\cos(2u-t)\right]_0^t$$

$$= \frac{1}{2}\mathrm{t}\sin \mathrm{t}$$

1.8 Applications of Laplace Transform

- Laplace transform is an effective way to solve ode, especially nonhomogeneous equation with input function in the form of special functions such as step or delta functions.
- This section discusses the implementation of Laplace transform in solving initial value problem (IVP) and boundary value problem (BVP).
- Thus, it will involve common derivatives.

Theorem 1.15 (Transforms of Derivatives)

$$\mathcal{L}\{y(t)\} = Y(s)$$

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0)$$

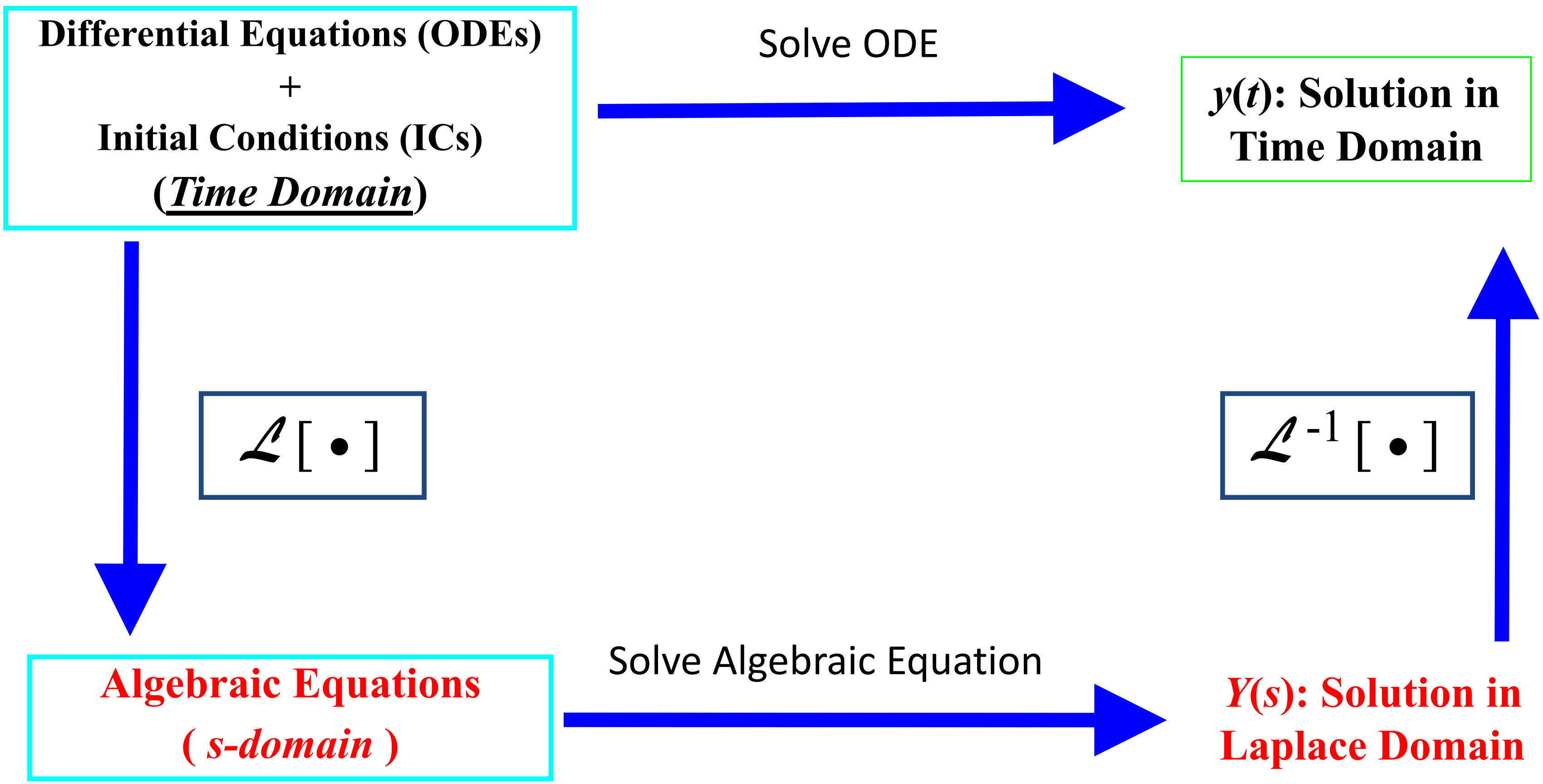
$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0)$$

$$\mathcal{L}\{y'''(t)\} = s^3Y(s) - s^2y(0) - sy'(0) - y''(0)$$

$$\mathcal{L}\{y^{(n)}t)\} = s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0)$$

1.8 Applications of Laplace Transform

Use Laplace Transform to solve IVP



Example 12

Solve the initial value problem

$$y' + y = \cos t\,, \qquad y(0) = 0$$

Solution

$$\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{\cos t\}$$

$$sY(s) - y(0) + Y(s) = \frac{s}{s^2 + 1}$$

By substituting initial condition $y(0) = 0$

$$sY(s) - 0 + Y(s) = \frac{s}{s^2 + 1}$$

$$(s + 1)Y(s) = \frac{s}{s^2 + 1}$$

$$Y(s) = \frac{s}{(s + 1)(s^2 + 1)}$$

$$\frac{s}{(s + 1)(s^2 + 1)} = \frac{A}{s + 1} + \frac{Bs + C}{s^2 + 1}$$

Use partial fraction to find the variables, A, B, C.

Example 12

By using partial fractions,

$$Y(s) = -\frac{1}{2(s+1)} + \frac{s}{2(s^2+1)} + \frac{1}{2(s^2+1)}$$

The solution $y(t)$ is obtained by taking the inverse of the Laplace transform.

$$y(t) = -\frac{1}{2}e^{-t} + \frac{1}{2}\cos t + \frac{1}{2}\sin t$$

Example 13

Solve the initial value problem

$$y'' + y = \sin 2t\,, \qquad y(0) = 2, \qquad y'(0) = 1$$

Solution:

Let $\mathcal{L}\{y(t)\} = Y(s)$. By using Laplace transform:

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{\sin 2t\}$$

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{2}{(s^2+4)}$$

Substitute the initial condition,

$$s^2Y(s) - s(2) - 1 + Y(s) = \frac{2}{(s^2+4)}$$

Example 13

$$s^2Y(s) - 2s - 1 + Y(s) = \frac{2}{(s^2+4)}$$

$$(s^2+1)Y(s) = \frac{2}{(s^2+4)} + 1 + 2s$$

$$Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)}$$

Using partial fractions, we can write $Y(s)$ in the form:

$$Y(s) = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

Example 13

By expanding the numerator of both sides:

$$2s^3 + s^2 + 8s + 6 = (a + c)\, s^3 + (b + d)s^2 + (4a + c)s + (4b + d)$$

Then by comparing the coefficients of like powers of s we have,

$$a + c = 2 \qquad b + d = 1$$
$$4a + c = 8 \qquad 4b + d = 6$$

Then we have, a$= 2, c = 0, b = \frac{5}{3}, d = -\frac{2}{3}$

Example 13

$$Y(s) = \frac{2s}{s^2+1} + \frac{5}{3}\left(\frac{1}{s^2+1}\right) - \frac{2}{3}\left(\frac{1}{s^2+4}\right)$$

The solution of given IVP is

$$y(t) = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t$$

Example 14

Solve the initial value problem:

$$y'' + 3y' + 2y = g(t),$$
$$y(0) = 0, y'(0) = 1$$
$$g(t) = \begin{cases} 1 & if\ 0 \le t < 1 \\ 0 & if\ \quad\quad t \ge 1 \end{cases}$$

Solution (Part 1)

$$g(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & t \ge 1 \end{cases}$$

g_1 -

Example 14

Solve the initial value problem:

$$y'' + 3y' + 2y = g(t),$$
$$y(0) = 0, y'(0) = 1$$
$$g(t) = \begin{cases} 1 & if\ 0 \le t < 1 \\ 0 & if\ \quad t \ge 1 \end{cases}$$

Solution (Part 2)

$$y'' + 3y' + 2y = H(t) - H(t-1)$$

$\mathcal{L}$

Example 14

Solve the initial value problem:

$$y'' + 3y' + 2y = g(t),$$
$$y(0) = 0, y'(0) = 1$$
$$g(t) = \begin{cases} 1 & if\ 0 \le t < 1 \\ 0 & if\ \quad\quad t \ge 1 \end{cases}$$

Solution (Part 3)

R11

Example 14

Solve the initial value problem:

$$y'' + 3y' + 2y = g(t),$$
$$y(0) = 0, y'(0) = 1$$
$$g(t) = \begin{cases} 1 & if\ 0 \leq t < 1 \\ 0 & if\ \quad\quad t \geq 1 \end{cases}$$

Solution (Part 4)

$$Y(s) = \frac{s+1}{s(s^2+3s+2)} - \frac{e^{-s}}{s(s^2+3s+2)}$$

yc'

Please attempt and I will upload the full solution soon

NEXT LESSON ? [WEBEX 3]

Chapter 1 (Part C)

Solving BVP by using Laplace transform.

Chapter 2 (Part A)

Introduction of Fourier Series.

Derivation of Fourier Series.

Fourier Series of Even and Odd Functions.

Thank You