JIM 319

Vector Calculus

Mohd. Asyraf Mansor

Differentiation of Scalar and Vector Functions

Vector Functions, Limit and Derivatives

Learning Objectives

At the end of this e-lecture, students should be able to:

- ❑ **Define the vector-valued functions and sketch space curves.**
- ❑ **Understand the concept of limit and continuity of vector functions.**
- ❑ **Compute the derivatives of vector functions.**
- ❑ **Understand the concept and properties of derivatives.**

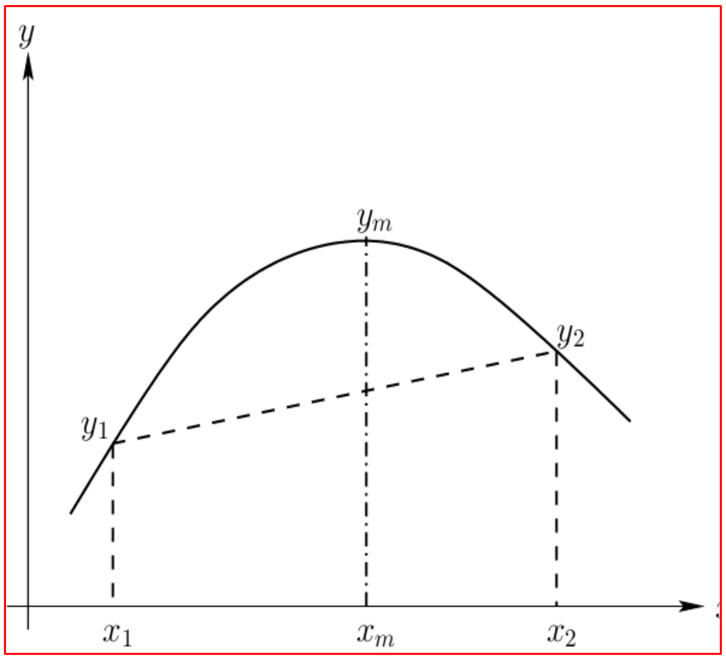
Introduction

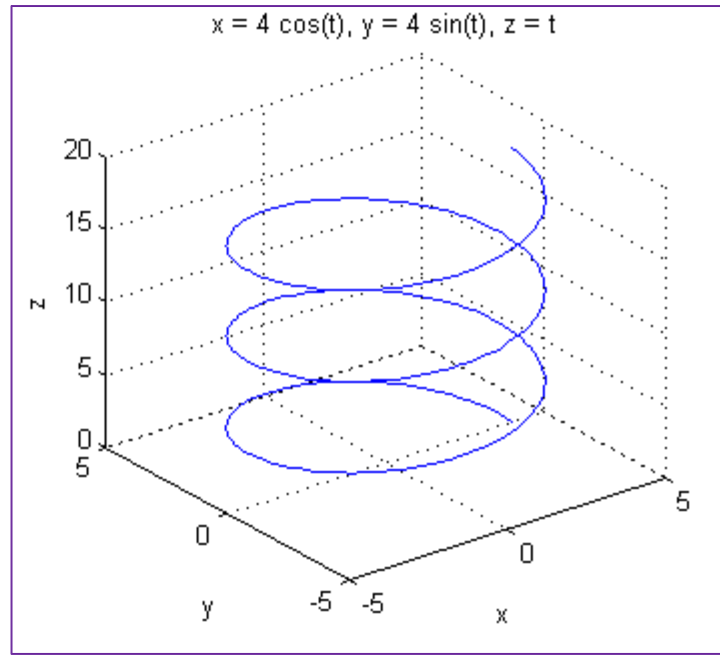
- In real calculus, we are dealing with real-valued function that definitely produced the scalar quantity.

- Meanwhile, in vector calculus, most of the problem will be in the form of vector-valued function.

Introduction

- How to represent **scalar-valued function** and **vector-valued function**?





Scalar-Valued Function

- A scalar-valued function is a function that takes one or more values but returns a single value. For example,

$$f(x,y,z) = x^2 + 2yz^5$$

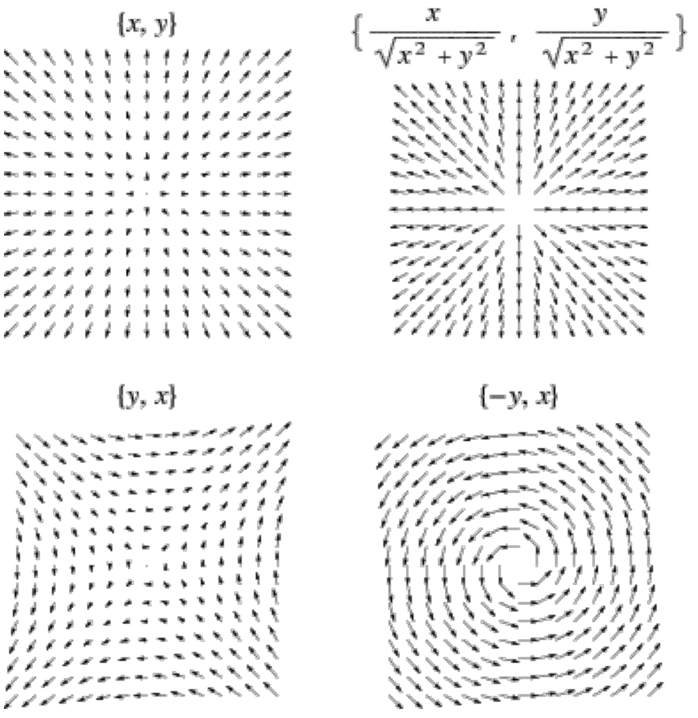
A n-variable scalar valued function acts as a map from the space **$\mathbf{R}^n$** to the real number line. That is, ***f*:$\mathbf{R}^n$->R**.

Scalar Field

- A "scalar field" is a fancy name for a function of space, i.e. it associates a real number with every position in some space, e.g. in 3D $\emptyset = \emptyset(x, y, z)$.
- Formally, scalar is a word used to distinguish the field from a vector field. If at every point in a region, a scalar function has a defined value, the region is called a **scalar field**.
- **Example**: Temperature distribution in a rod.

Vector Field

- A vector field is one where a quantity in "space" is represented by both magnitude and direction, i.e by vectors.
- Imagine rain on the mountain. The vectors are also "streamlines." Water running down the mountain will follow these streamlines.



Vector-Valued Function

- In general, a function can be defined as a rule that assigns to each element in domain and range.
- A vector-valued function or vector function, is a function whose domain comprises of a set of real numbers and the range is a set of vectors.
- In this section, our focus will be the vector functions $\underline{r}$ whose are $\mathbb{R}^3$ vectors.

Example 1

Determine the domain of the following function:

$$\underline{r} = (\cos t\,, \ln(4-t)\,, \sqrt{t+1})$$

The first component is defined for all t's.

The second component is only defined for $t < 4$.

The third component is only defined for $t \geq -1$.

Putting all of these together gives the following domain.

$$\text{Domain of } \underline{r} = [-1,4)$$

Example 2

Sketch the graph of each of the following vector functions.

$$\underline{r}(t) = (t, t^3 - 10t + 7)$$

Solution

Try a few evaluation for this vector function,

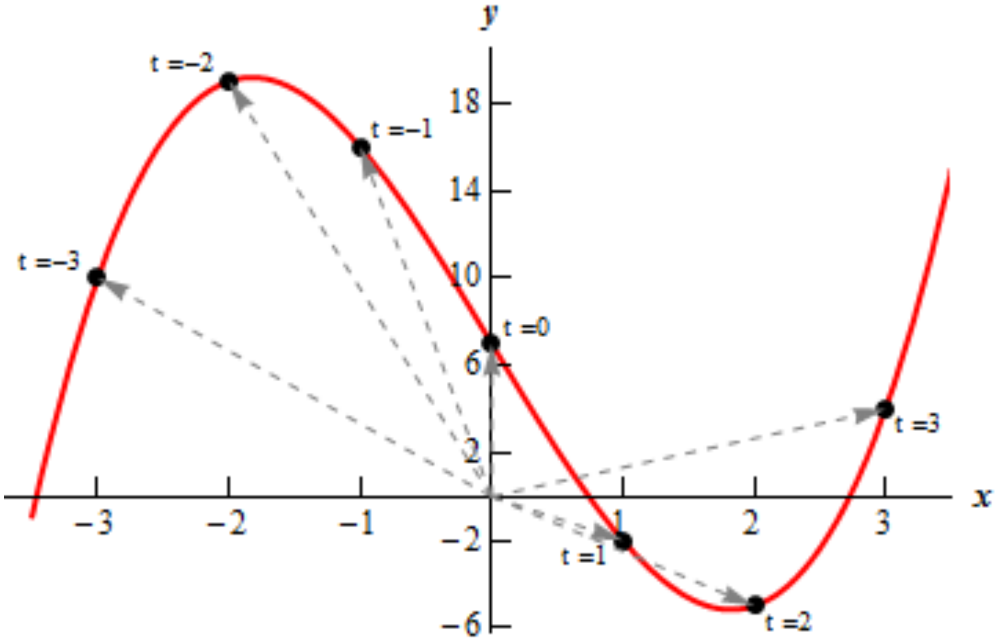
$$\underline{r}(-3) = (-3,10), \qquad \underline{r}(-1) = (-1,16)$$

$$\underline{r}(1) = (1,-2), \qquad \underline{r}(3) = (3,4)$$

So, we've got a few points on the graph of this function. In general, it can take quite a few function evaluations to get an idea of what the graph is and it's usually easier to use a computer to do the graphing.

Example 2

Therefore, we can sketch the graph for that vector function. We've put in a few vectors/evaluations to illustrate them, but the reality is that we did have to use a computer to get a good sketch here.



Example 3

Sketch the graph of the following vector function.

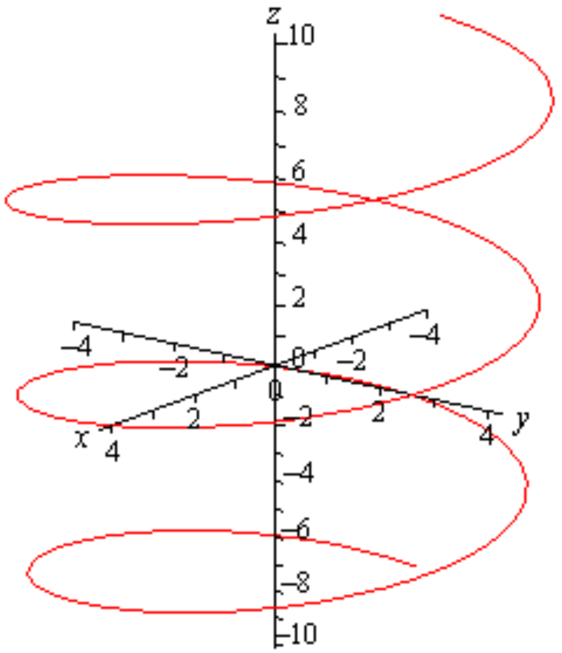
$$\underline{r}(t) = (4\cos t, 4\sin t\,, t)$$

Solution:

- If this one had a constant in the z component we would have another circle.
- However, in this case we don't have a constant. Instead we've got a t and that will change the curve.
- However, because the x and y component functions are still a circle in parametric equations our curve should have a circular nature to it in some way.

Example 3

- In fact, the only change is in the z component and as t increases the z coordinate will increase.
- Also, as t increases the x and y coordinates will continue to form a circle centered on the z-axis. Here is a sketch of this curve.



Limit and Continuity of Vector Functions

- The limit of a vector function $\underline{r}$ is defined by taking the limits of its component functions as follows:

If $\underline{r}(t) = f(t)\underline{i} + g(t)\underline{j} + h(t)\underline{k}$, then

$$\lim_{t \to a} \underline{r}(t) = [\lim_{t \to a} f(t)]\, \underline{i} + [\lim_{t \to a} f(t)\, g(t)]\, \underline{j} + [\lim_{t \to a} f(t)\, h(t)]\, \underline{k}$$

provided the limits of the component function exists.

Limit and Continuity of Vector Functions

- Limits of vector functions obey the same rules as limits of real-valued functions.

Example 4

Find the $\lim_{t \to 0} \underline{r}(t) = (1 + t^3)\underline{i} + te^{-t}\underline{j} + \frac{\sin t}{t}\underline{k}$

Solution:

According to definition,

$$\lim_{t \to 0} \underline{r}(t) = [\lim_{t \to 0}(1 + t^3)]\,\underline{i} + [\lim_{t \to 0} te^{-t}]\,\underline{j}$$

$$+ [\lim_{t \to 0} \frac{\sin t}{t}]\underline{k}$$

$$= \underline{i} + \underline{k}$$

Limit and Continuity of Vector Functions

- A vector function $\underline{r}$ is continuous at $t=a$ if

I. $\lim_{t \to a} \underline{r}(t)$ exists

II. $\underline{r}$ is defined at a.

III. $\lim_{t \to a} \underline{r}(t) = \underline{r}(a)$

Derivatives of Vector Functions

- The derivatives $\underline{r}'$ of a vector function $\underline{r}$ *is* defined in the same way as for real-valued functions:
- Given the following equation:

$$\frac{d\underline{r}}{dt} = \underline{r}'(t) = \lim_{h \to 0} \frac{\underline{r}(t+h) - \underline{r}(t)}{h}$$

Example 5

Find the derivative of $\underline{r}(t) = (1 + t^3)\underline{i} + (te^{-t})\underline{j} + \sin 2t\,\underline{k}$. Hence, find the unit tangent vector at the point where $t = 0$.

SOLUTION:

$$\frac{d\underline{r}}{dt} = \underline{r'}(t) = \frac{d}{dt}((1 + t^3)\underline{i} + (te^{-t})\underline{j} + \sin 2t\,\underline{k})$$

$$= 3t^2\,\underline{i} + (1 - \mathrm{t})\,e^{-t}\,\underline{j} + 2\cos 2t\,\underline{k}$$

Example 5

when $t = 0$.

$$\underline{r}'(t) = 3(0)^2\,\underline{i} + (1-0)\,e^{-0}\,\underline{j} + 2\cos 2(0)\,\underline{k}$$

$$= \underline{j} + 2\,\underline{k}$$

$$\|\underline{r}'(t)\| = \sqrt{(1)^2 + (2)^2} = \sqrt{5}$$

Unit tangent vector,

$$\underline{T} = \frac{\underline{r}'(t)}{\|\underline{r}'(t)\|} = \frac{1}{\sqrt{5}}\underline{j} + \frac{2}{\sqrt{5}}\underline{k}$$

Derivatives of Vector Functions

- Suppose $\underline{A}$, $\underline{B}$, and $\underline{C}$ are differentiable vector function a of a scalar u and ϕ is a differentiable scalar function of u. The following law hold:

LAW 1 (Addition)

$$\frac{d}{du}(\underline{A}+\underline{B}) = \frac{d\underline{A}}{du} + \frac{d\underline{B}}{du}$$

Derivatives of Vector Functions

LAW 2 (Product Rule)

$$\frac{d}{du}\left(\underline{A}.\underline{B}\right) = \underline{A}.\frac{d\underline{B}}{du} + \frac{d\underline{A}}{du}.\underline{B}$$

LAW 3 (Cross Product)

$$\frac{d}{du}\left(\underline{A}\times\underline{B}\right) = \underline{A}\times\frac{d\underline{B}}{du} + \frac{d\underline{A}}{du}\times\underline{B}$$

LAW 4 (Product Rule)

$$\frac{d}{du}\left(\phi\underline{A}\right) = \phi\frac{d\underline{A}}{du} + \frac{d\phi}{du}\underline{A}$$

Derivatives of Vector Functions

LAW 5 (Scalar Triple Product)

$$\frac{d}{du}(\underline{A}\,.\,\underline{B}\times\underline{C}) = \underline{A}\,.\,\underline{B}\times\frac{d\underline{C}}{du} + \underline{A}\,.\,\frac{d\underline{B}}{du}\times\underline{C} + \frac{d\underline{A}}{du}.\,\underline{B}\times\underline{C}$$

LAW 6 (Vector Triple Product)

$$\frac{d}{du}\{\underline{A}\times(\underline{B}\times\underline{C})\} = \underline{A}\times\left(\underline{B}\times\frac{d\underline{C}}{du}\right) + \underline{A}\times\left(\frac{d\underline{B}}{du}\times\underline{C}\right) + \frac{d\underline{A}}{du}\times\left(\underline{B}\times\underline{C}\right)$$

Example 6 (a)

Suppose $\underline{A} = 5u^2\underline{i} + u\,\underline{j} - u^3\underline{k}$ and $\underline{B} = \sin u\;\underline{i} - \cos u\;\underline{j}$. Find $\frac{d}{du}\left(\underline{A}.\underline{B}\right)$.

Solution:

$$\frac{d}{du}\left(\underline{A}.\underline{B}\right) = \underline{A}.\frac{d\underline{\underline{B}}}{du} + \frac{d\underline{\underline{A}}}{du}.\underline{B}$$

$$\frac{d\underline{\underline{B}}}{du} = \cos u\;\;\underline{i} + \sin u\;\;\underline{j} \qquad \frac{d\underline{\underline{A}}}{du} = 10u\,\underline{i} + \underline{j} - 3u^2\underline{k}$$

$$= (5u^2\underline{i} + u\,\underline{j} - u^3\underline{k}).(\cos u\,\underline{i} + \sin u\,\underline{j}) + \left(10u\,\underline{i} + \underline{j} - 3u^2\underline{k}\right).(\sin u\;\;\underline{i} - \cos u\;\;\underline{j})$$

$$= [5u^2\,cos\,u + u\,sin\,u] + [10\,u\;\;sin\,u - cos\,u]$$

$$= (5u^2 - 1)\,cos\,u + 11u\,sin\,u$$

Example 6 (a)

Alternative Method:

$$\underline{A}.\underline{B} = 5u^2 \sin u - u \cos u$$

$$\frac{d}{du}\left(\underline{A}.\underline{B}\right) = \frac{d}{du}(5u^2 \sin u - u \cos u)$$

$$= 5u^2 \cos u + 10u \sin u + u \sin u - \cos u$$

$$= (5u^2 - 1) \cos u + 11u \sin u$$

Example 6 (b)

Find $\frac{d}{du}(\underline{A} \times \underline{B})$.

Solution:

$$\frac{d}{du}(\underline{A} \times \underline{B}) = \underline{A} \times \frac{d\underline{B}}{du} + \frac{d\underline{A}}{du} \times \underline{B}$$

$$\frac{d\underline{B}}{du} = \cos u \ \underline{i} + \sin u \ \underline{j}$$

$$\frac{d\underline{A}}{du} = 10u \ \underline{i} + \underline{j} - 3u^2\underline{k}$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 5u^2 & u & -u^3 \\ \cos u & \sin u & 0 \end{vmatrix} + \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 10u & 1 & -3u^2 \\ \sin u & -\cos u & 0 \end{vmatrix}$$

Example 6 (b)

$$\underline{A} \times \frac{d\underline{B}}{du} + \frac{d\underline{A}}{du} \times \underline{B} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 5u^2 & u & -u^3 \\ \cos u & \sin u & 0 \end{vmatrix} + \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 10u & 1 & -3u^2 \\ \sin u & -\cos u & 0 \end{vmatrix}$$

$= [u^3 \sin u\ \underline{i} - u^3\ \cos u\ \underline{j} + (5u^2 \sin u - u \cos u)\underline{k}\ ] +$
$[-3u^2 \cos u\ \underline{i} - 3u^2 \sin u\ \underline{j} + (-10u \cos u - \sin u)\ \underline{k}\ ]$

$= (u^3 \sin u - 3u^2 \cos u)\ \underline{i} - (u^3\ \cos u + 3u^2\ \sin u)\ \underline{j} + (5u^2$
$\sin u - \sin u - 11u\ \cos u)\underline{k}$

Example 6 (b)

Alternative Method:

$$\underline{A}\times\underline{B} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 5u^2 & u & -u^3 \\ \sin u & -\cos u & 0 \end{vmatrix}$$

$$= -u^3 \cos u\, \underline{i} - u^3 \sin u\, \underline{j} + (-5u^2 \cos u - u \sin u)\, \underline{k}$$

$$\frac{d}{du}(\underline{A}\times\underline{B}) = \frac{d}{du}(-u^3 \cos u)\, \underline{i} - \frac{d}{du}(u^3 \sin u)\, \underline{j} + \frac{d}{du}(-5u^2 \cos u - u \sin u)\, \underline{k}$$

$$= (u^3 \sin u - 3u^2 \cos u)\, \underline{i} - (u^3 \cos u + 3u^2 \sin u)\, \underline{j}$$
$$+ (5u^2 \sin u - \sin u - 11u \cos u)\underline{k}$$

Example 6 (c)

Suppose $\underline{A} = 5u^2\underline{i} + u\,\underline{j} - u^3\underline{k}$ and $\underline{B} = \sin u\,\underline{i} - \cos u\,\underline{j}$. Find $\frac{d}{du}\left(\underline{A}.\underline{A}\right)$.

Solution:

$$\frac{d}{du}\left(\underline{A}.\underline{A}\right) = \underline{A}.\frac{d\underline{A}}{du} + \frac{d\underline{A}}{du}.\underline{A} = 2\,\underline{A}.\frac{d\underline{A}}{du}$$

$$\frac{d\underline{A}}{du} = 10u\,\underline{i} + \underline{j} - 3u^2\underline{k}$$

$$= 2\left(5u^2\underline{i} + u\,\underline{j} - u^3\underline{k}\right).(10u\,\underline{i} + \underline{j} - 3u^2\underline{k})$$

$$= 100u^3 + 2u + 6u^5$$

Example 6 (c)

Alternative Method:

$$\underline{A}.\underline{A} = (5u^2)^2 + (u)^2 + (-u^3)^2$$

$$= 25u^4 + u^2 + u^6$$

$$\frac{d}{du}(\underline{A}.\underline{A}) = \frac{d}{du}(25u^4 + u^2 + u^6)$$

$$= 100u^3 + 2u + 6u^5$$

Example 7 (a)-(d)

Given $\underline{R} = (3\cos t)\underline{i} + (3\sin t)\underline{j} + (4t)\underline{k}$.

Find:

(a) $\frac{d\underline{R}}{dt}$

(b) $\frac{d^2\underline{R}}{dt^2}$

(c) $\left\|\frac{d\underline{R}}{dt}\right\|$

(d) $\left\|\frac{d^2\underline{R}}{dt^2}\right\|$

Example 7 (a)-(d)

(a) $\frac{d\underline{\underline{R}}}{dt} = \frac{d}{dt}(3\cos t)\underline{i} + \frac{d}{dt}(3\sin t)\underline{j} + \frac{d}{dt}(4t)\underline{k}$

$= (-3\sin t)\,\underline{i} + (3\cos t)\,\underline{j} + 4\,\underline{k}$

(b) $\frac{d^2\underline{\underline{R}}}{dt^2} =$

$= \frac{d}{dt}\left(\frac{dR}{dt}\right) = \frac{d}{dt}(-3\sin t)\,\underline{i} + \frac{d}{dt}(3\cos t)\,\underline{j} + \frac{d}{dt}(4)\,\underline{k}$

$= (-3\cos t)\,\underline{i} + (-3\sin t)\,\underline{j}$

$= (-3\cos t)\,\underline{i} - (3\sin t)\,\underline{j}$

Example 7 (a)-(d)

(c) $\left\|\frac{d\underline{R}}{dt}\right\|$

$$= \sqrt{(-3\sin t)^2+(3\cos t)^2+4^2}$$

$$= \sqrt{9\sin^2 t + 9\cos^2 t + 16} = \sqrt{25} = 5$$

(d) $\left\|\frac{d^2\underline{R}}{dt^2}\right\|$

$$= \sqrt{(-3\cos t)^2+(-3\sin t)^2+0^2}$$

$$= \sqrt{9(\sin^2 t + \cos^2 t)} = \sqrt{9} = 3$$

Example 8

Suppose $\underline{A}$ has constant magnitude. Show that $\underline{A}.\frac{d\underline{\underline{A}}}{dt} = 0$ and that $\underline{A}$ and $\frac{d\underline{\underline{A}}}{dt}$ are perpendicular provided $\left\|\frac{d\underline{\underline{A}}}{dt}\right\| \neq 0.$

$$\frac{d}{dt}(\underline{A}.\underline{A}) = \underline{A}.\frac{d\underline{\underline{A}}}{dt} + \frac{d\underline{\underline{A}}}{dt}.\underline{A}$$

$$= 2\underline{A}.\frac{d\underline{\underline{A}}}{dt} = 0$$

Thus, $\underline{A}.\frac{d\underline{\underline{A}}}{dt} = 0$ and $\underline{A}$ is perpendicular to $\frac{d\underline{\underline{A}}}{dt}$ provided $\left\|\frac{d\underline{\underline{A}}}{dt}\right\| \neq 0.$

Example 9

Suppose $\underline{p} = 3t\,\underline{i} - t^2\underline{j}$ and $\underline{q} = 2t^2\,\underline{i} + 3\underline{j}$. Verify the result $\frac{d}{dt}(\underline{p}.\,\underline{q}) = \underline{p}\,.\frac{d\underline{q}}{du} + \frac{d\underline{p}}{du}.\,\underline{q}$

Solution:

$$\underline{p}.\,\underline{q} = (3t)(2t^2) - (3t^2) = 6t^3 - 3t^2$$

$$\frac{d}{dt}(\underline{p}.\,\underline{q}) = 18t^2 - 6t$$

Example 9

$$\frac{d\underline{p}}{du} = 3\underline{i} - 2t\underline{j} \qquad \frac{d\underline{q}}{du} = 4t\ \underline{i}$$

$$\underline{p}\cdot\frac{d\underline{q}}{du} + \frac{d\underline{p}}{du}\cdot\underline{q} = \left(3t\ \underline{i} - t^2\underline{j}\right).(4t\ \underline{i}) + \left(3\underline{i} - 2t\underline{j}\right).\left(2t^2\ \underline{i} + 3\underline{j}\right)$$

$$= 12t^2 + 6t^2 - 6t$$

$$= 18\ t^2 - 6t$$

We have verified $\frac{d}{dt}(\underline{p}.\ \underline{q}) = \underline{p}\cdot\frac{d\underline{q}}{du} + \frac{d\underline{p}}{du}\cdot\underline{q}$

Example 10

If $\underline{r}(t) = 4t^2\underline{i} + 2t\,\underline{j} - 7\underline{k}$, evaluate $\underline{r}(t)$ and $\underline{r}'(t)$ when $t = 1$.

$$\underline{r}(t) = 4(1)^2\underline{i} + 2(1)\underline{j} - 7\underline{k}$$
$$= 4\underline{i} + 2\underline{j} - 7\underline{k}$$

$$\underline{r}'(t) = 8t\,\underline{i} + 2\,\underline{j} = 8(1)\,\underline{i} + \ 2\,\underline{j}$$
$$= 8\,\underline{i} + \ 2\,\underline{j}$$

Thank You