JIM 319

Vector Calculus

WEBEX Class 1:
Vectors, Vector Algebra and Application

14th September 2019

10.00-11.00 am (Saturday)

Lecturer: Dr. Mohd. Asyraf Mansor

asyrafman@usm.my

Learning Objectives

At the end of this WEBEX class, students should be able to:

- ❑ **Understand the fundamental concept of vector and vector algebra.**
- ❑ **Compute dot (scalar) product of two vectors.**
- ❑ **Compute the cross (vector) product of two vectors.**
- ❑ **Compute triple scalar product of three vectors.**

INTRODUCE YOURSELF [COMPULSORY]

Welcome to the new semester! We hope that we have compiled an interesting course for you. Our first WEBEX class will be held on 14th September 2019 (Saturday). Before we meet, you are compulsory to complete the following:

You are required to introduce yourself in the following Padlet. Click the link below:

JIM319 PADLET LINK

Your padlet must contain the following details

(i) Your personal picture. Group photo is not allowed.

(ii) Your Simple Autobiography (Hometown, Hobby, Current Job or anything related)

(iii) What do you want to achieve in this course?

JIM319 PADLET PREVIEW

WEBEX CLASS 1

(E-LECTURE 1.1-1.3)

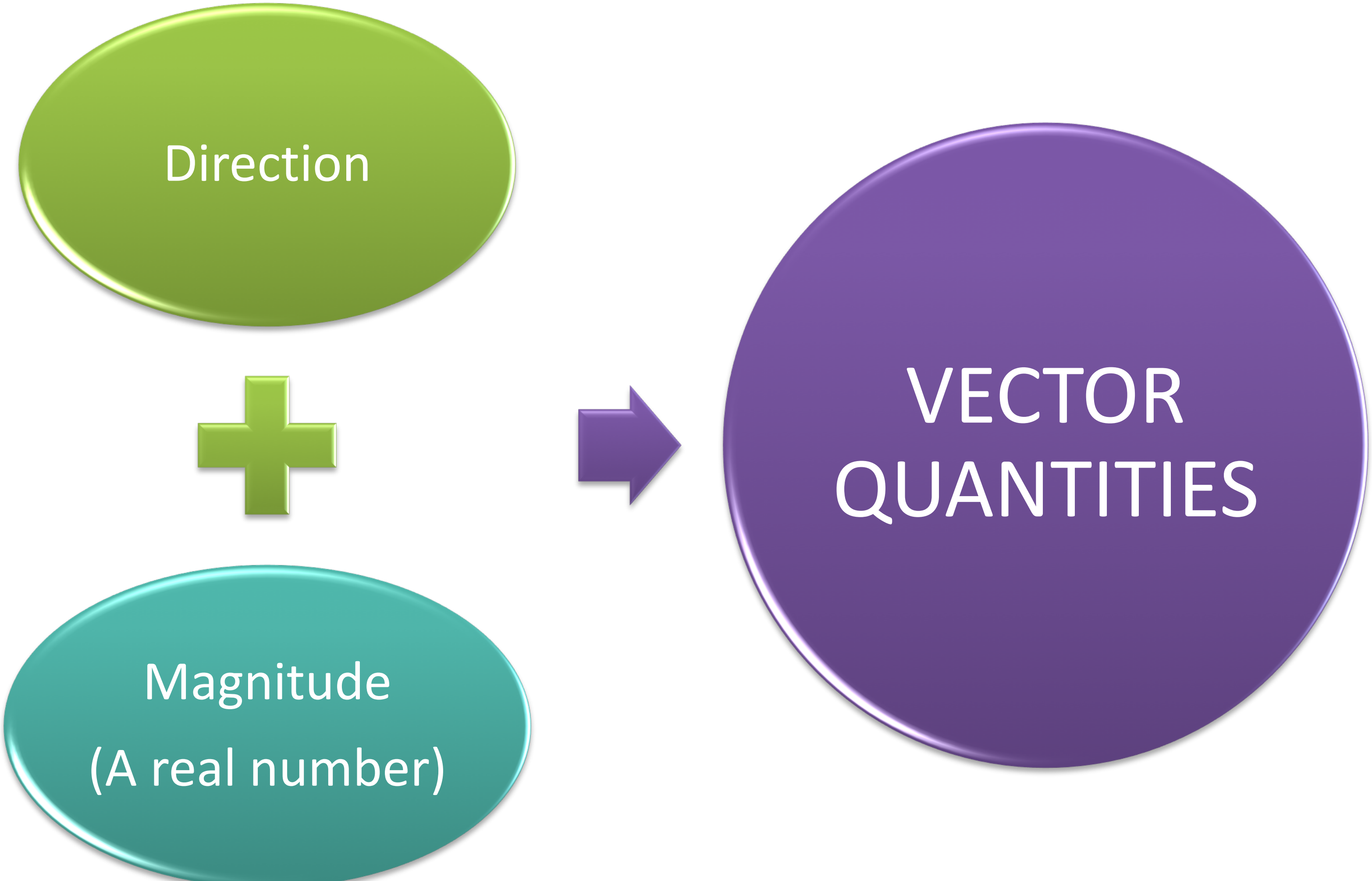
Scalar and Vector Quantities

- **What do you call quantities that can be described completely by a single number (with appropriate unit) ?**

Give some examples…..

- **Physical quantities that require number and direction to describe them completely are called…**

Give some examples…..


Direction
Magnitude
(A real number)
VECTOR QUANTITIES

Scalar and Vector Quantities

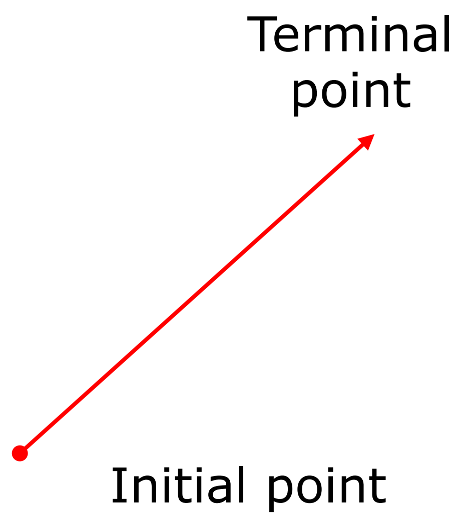
A **scalar quantity** has only **magnitude.**

A **vector quantity** has both **magnitude** and **direction.**

Scalar Quantities	Vector Quantities
length, area, volume	displacement
speed	velocity
mass, density	acceleration
pressure	momentum
temperature	force
energy, entropy	lift , drag , thrust
work, power	weight

What is a Vector?

- A quantity that has <u>both</u>
  - Size
  - Direction
- Examples
  - Wind
  - Boat or aircraft travel
  - Forces in physics
- Geometrically
  - A directed line segment



How do we represent a Vector?

Two general approaches to represent vector :

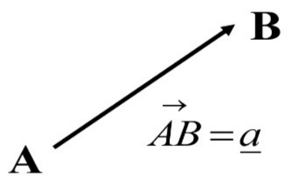
Geometric approach

- Drawing line elements in space (Euclidean geometry).

Coordinate (Analytical) approach

- Assumes that space is defined by Cartesian coordinates, and uses these to characterize vectors

What is a Vector?

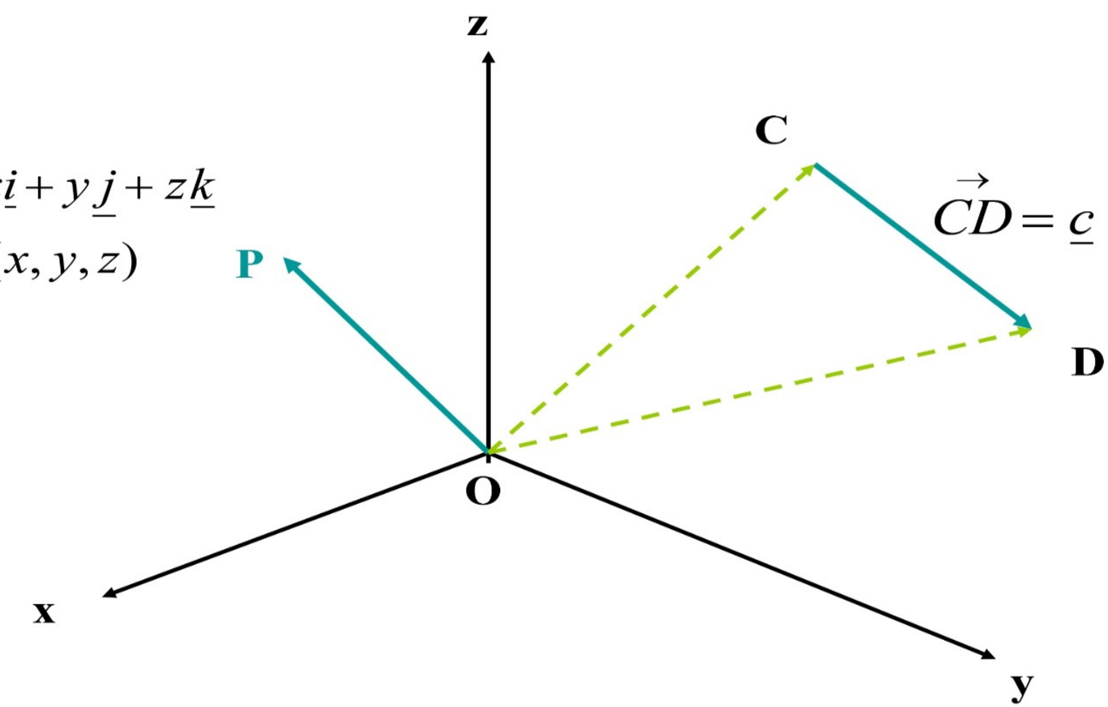


$\underline{a}$ is a free vector

NOT refered to any

frame of reference

$$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$$
$$= (x, y, z)$$



Vector Algebra

Given real numbers

$$x, y, z \in R$$

compute

$$x \pm y, \quad x \times y$$

$$\frac{x}{y}$$

Real Algebra

Given real number

$$x \in R$$

and vectors

$$\underline{a}, \underline{b} \in R^3$$

compute

$$\underline{a} \pm \underline{b},$$

$$x\underline{a}, \quad \underline{a} . \underline{b}, \quad \underline{a} \times \underline{b}$$

Vector Algebra

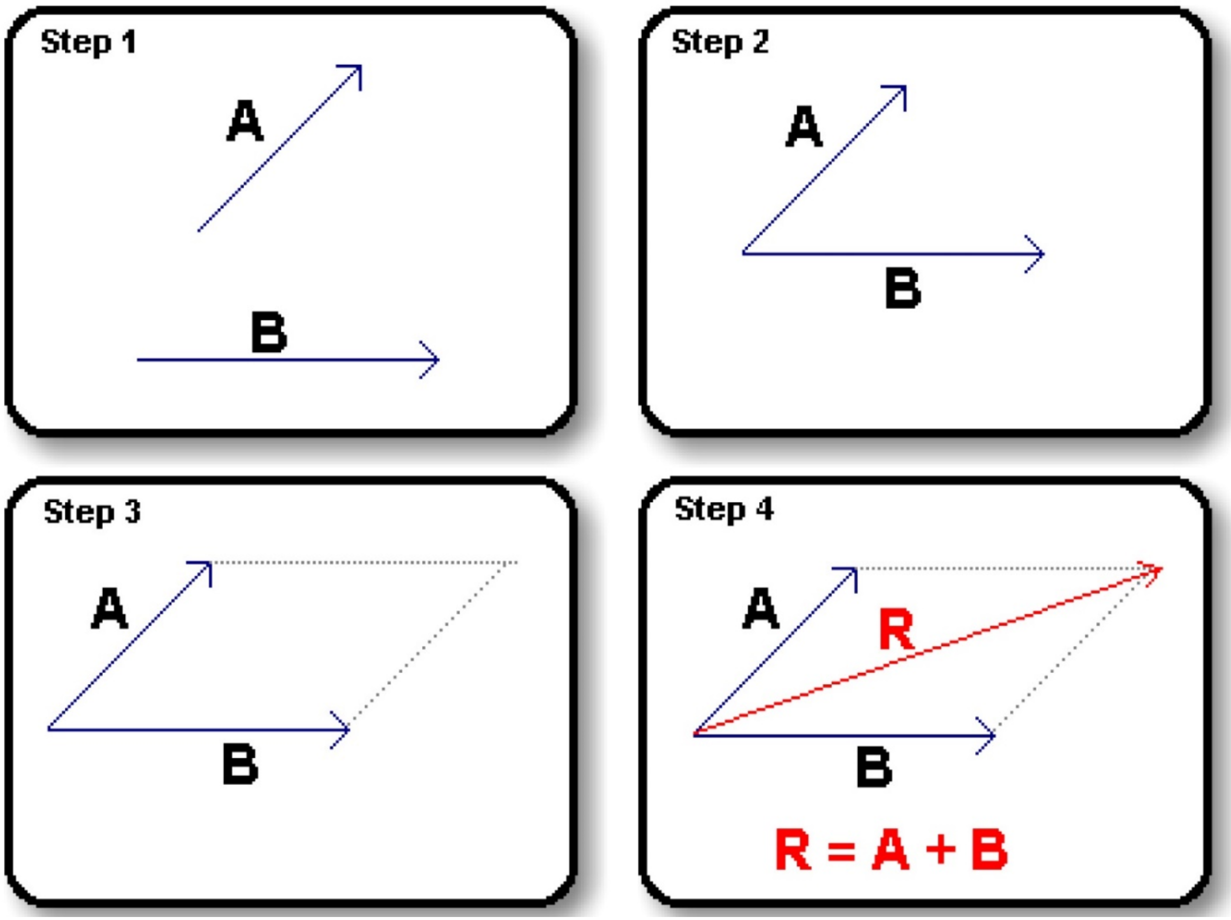
Scalar	Vector
$x \pm y$	$\underline{a} \pm \underline{b}$
βx	$\beta \underline{a}$
	$\underline{a} . \underline{b}$
	$\underline{a} \times \underline{b}$
$\frac{x}{y}, y \neq 0$	$\frac{\underline{a}}{\underline{b}}$?

Scalar Multiplication

Scalar/Dot Product

Vector/Cross Product

Vector Addition: Parallelogram Law



Magnitude of a Vector

The magnitude of the vector is written as $\|\underline{A}\|$.

It can be deduced from the concept of Phytagoras's theorem that the magnitude of the vector can be written as its components as

$$\|\underline{A}\| = \sqrt{A_1^2 + A_2^2 + A_3^2} \quad \text{(for vector in } \mathbb{R}^3\text{).}$$

WORKED EXAMPLE 1

Find the magnitude of vector $\underline{z} = -2\underline{i} + 1\underline{j} + 2\underline{k}$.

$$\begin{aligned}\|\underline{z}\| &= \sqrt{(-2)^2+(1)^2+(2)^2} \\ &= \sqrt{9} \\ &= 3\end{aligned}$$

Unit Vectors

Unit vectors are vectors having unit length. The formula of unit vector is $\frac{\underline{z}}{\|\underline{z}\|}$

WORKED EXAMPLE 2

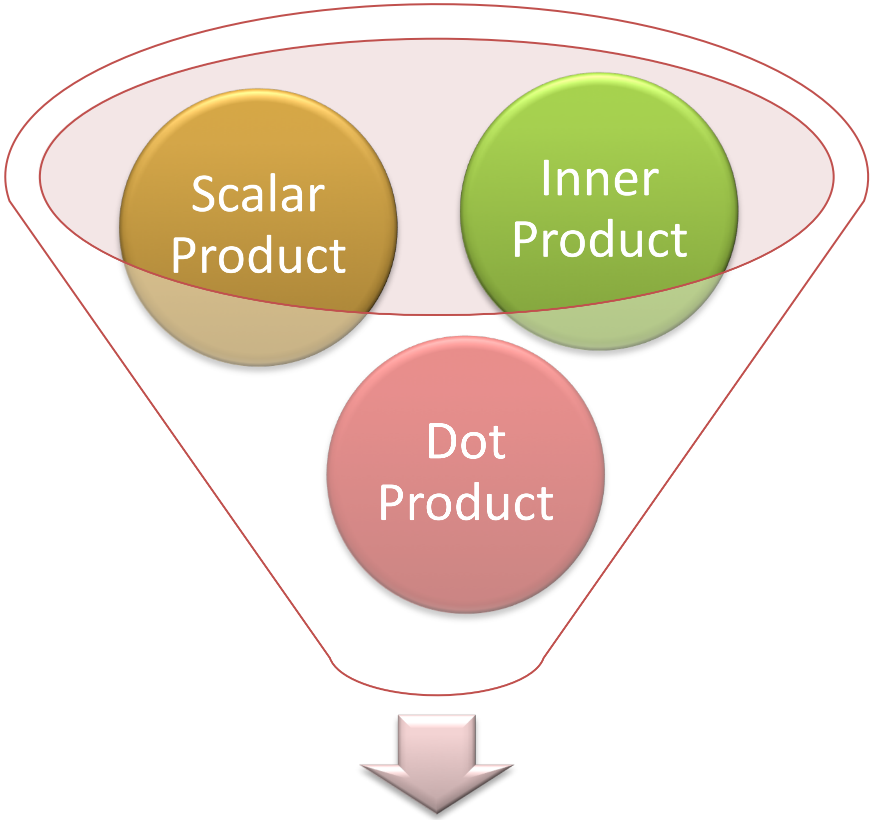
Find the unit vector in direction of $\underline{z} = \underline{i} - 2\underline{j} + \underline{k}$

The magnitude of $\|\underline{z}\| = \sqrt{(1)^2+(-2)^2+(1)^2}$

$= \sqrt{6}$

Therefore, the unit vector $\frac{\underline{z}}{\|\underline{z}\|} = \frac{1}{\sqrt{6}}\underline{i} - \frac{2}{\sqrt{6}}\underline{j} + \frac{1}{\sqrt{6}}\underline{k}$

The Dot Product



Dot product of two vectors is a *real number*.

Algebraic Version of Dot Product

- Given $\underline{a}, \underline{b} \in \mathbb{R}^2$, then

$$\underline{a} \cdot \underline{b} = \left(a_1\underline{i} + a_2\underline{j}\right) \cdot \left(b_1\underline{i} + b_2\underline{j}\right)$$
$$= a_1\, b_1 + a_2\, b_2$$

- Let $\underline{a}$ and $\underline{b}$ are vectors in $\mathbb{R}^3$,

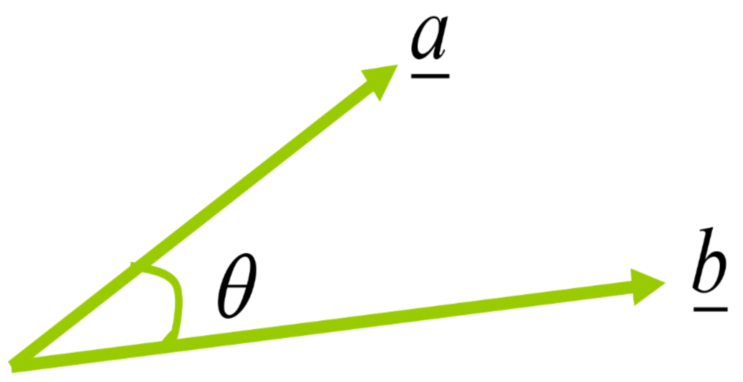
$$\underline{a} \cdot \underline{b} = \left(a_1\underline{i} + a_2\underline{j} + a_3\underline{k}\right) \cdot \left(b_1\underline{i} + b_2\underline{j} + b_3\underline{k}\right)$$
$$= a_1\, b_1 + a_2\, b_2 + a_3\, b_3$$

Geometric Version of Dot Product

Let $\underline{a}$ and $\underline{b}$ be nonzero vector in $\mathbb{R}^2$ *or* $\mathbb{R}^3$. Then,

$$\underline{a} \cdot \underline{b} = \left\|\underline{a}\right\| \left\|\underline{b}\right\| \cos\theta$$

where θ is the angle between $\underline{a}$ and $\underline{b}$.



Test for Orthogonality of Vectors

The terms ***orthogonal*** and ***perpendicular*** mean essentially the same thing—meeting at right angles.

DEFINITION : Vectors $\underline{a}\ and\ \underline{b}$ are orthogonal if $\underline{a}\,.\,\underline{b} = 0$

Even though the angle between the zero vector and another vector is not defined, it is convenient to extend the definition of orthogonality to include the zero vector.

In other words, the zero vector is orthogonal to every vector $\underline{a}$ because $\mathbf{0} \cdot \underline{a} = 0$.

The Algebraic Definition of the Cross Product

Let $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$ be vector in $\mathbb{R}^3$. The cross product of $\underline{a}\times\underline{b}$ is the vector:

$$\underline{a}\times\underline{b} = (a_2b_3 - a_3b_2)\underline{i} - (a_1b_3 - a_3b_1)\underline{j} + (a_1b_2 - a_2b_1)\underline{k}$$

The Algebraic Definition of the Cross Product

Similarly, the cross product of $\underline{a} \times \underline{b}$ can be computed via determinant method.

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \underline{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \underline{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \underline{k}$$

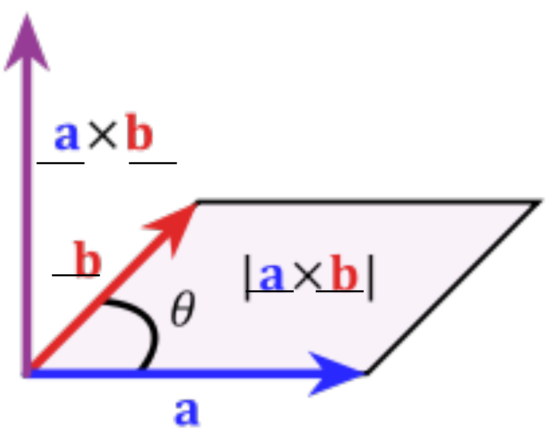
$$= (a_2 b_3 - a_3 b_2)\underline{i} - (a_1 b_3 - a_3 b_1)\underline{j} + (a_1 b_2 - a_2 b_1)\underline{k}$$

Which is precisely the definition of the cross product !!

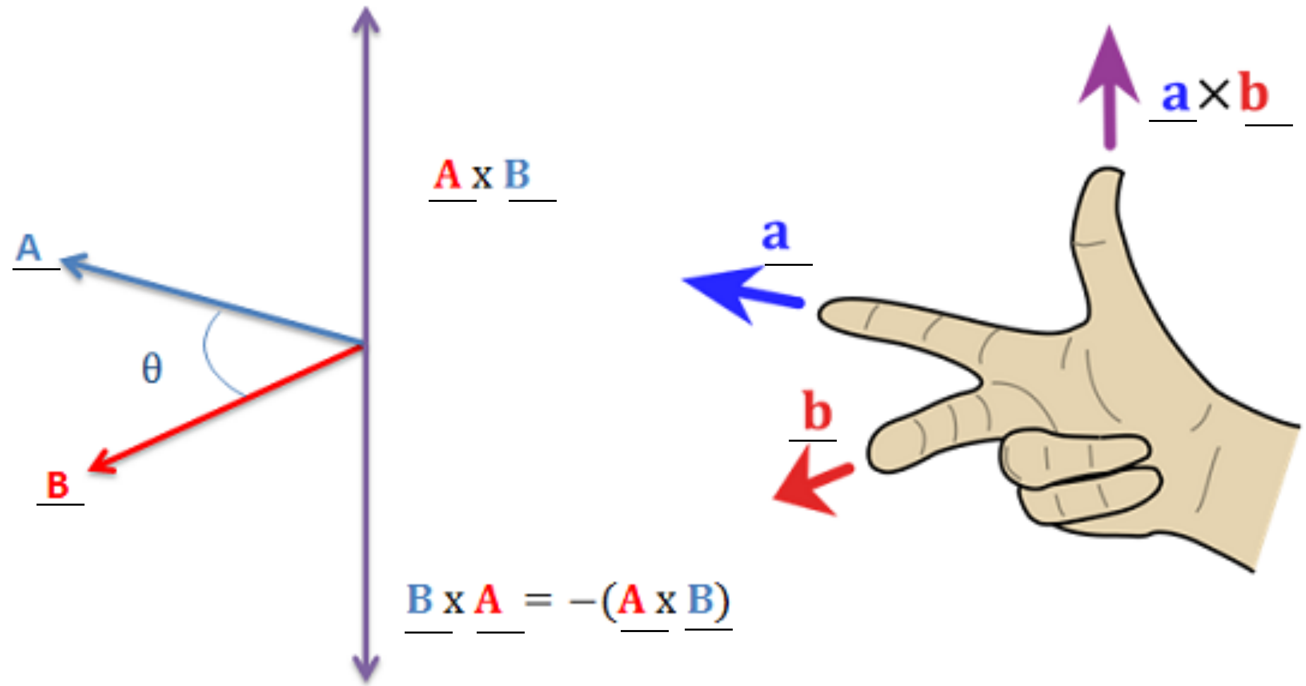
The Geometric Definition of the Cross Product

Let $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$ be non-zero vector in $\mathbb{R}^3$ and θ be the angle between $\underline{a}$ and $\underline{b}$. Geometrically speaking, the cross product of $\underline{a}\times\underline{b}$ is the vector:

$$\|\underline{a}\times\underline{b}\| = \|\underline{a}\|\,\|\underline{b}\|\sin\theta$$



The Geometric Definition of the Cross Product



Scalar Triple Product

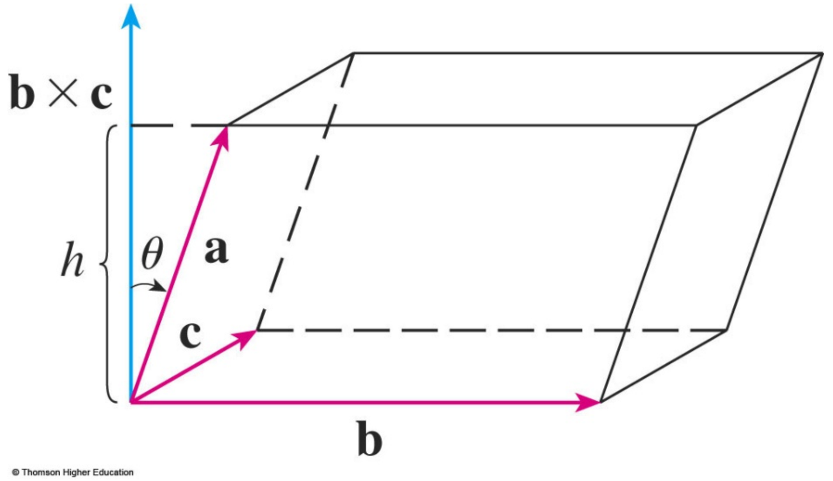
- The product $\underline{a}\,.\,(\underline{b}\times\underline{c})$ called the scalar triple product of the vectors $\underline{a}$, $\underline{b}$, and $\underline{c}$.
- It is also known as the box product.
- We can write the scalar triple product as a determinant:

$$\underline{\mathbf{a}}\cdot(\underline{\mathbf{b}}\times\underline{\mathbf{c}})=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Scalar Triple Product: Volume of the Parallelepiped

- The volume of the parallelepiped determined by the vectors $\underline{a}$, $\underline{b}$, and $\underline{c}$ is the magnitude of their scalar triple product:

$$V = | \underline{a} \cdot (\underline{b} \times \underline{c}) |$$



Coplanar Vectors

- If the volume of the parallelepiped determined by $\underline{a}$, $\underline{b}$, and $\underline{c}$ is 0, then the vectors must lie in the same plane.

$$\left|\underline{a} \,.\, (\underline{b} \times \underline{c})\right| = 0$$

  - That is, they are coplanar.

WEBEX CLASS 1

EXERCISES AND TYPICAL EXAMINATION QUESTIONS

Question 1

State which of the following are scalars and which are vectors:

(a) Specific heat

Scalar

(b) Momentum

Vector

(c) Magnetic Field Intensity

Vector

(d) Speed

Scalar

Question 2:
Given $\underline{a} = 2\underline{i} - 5\underline{j} + \underline{k}$ and $\underline{b} = 4\underline{i} - 2\underline{j} - 9\underline{k}$. Find $\underline{a} \cdot \underline{b}$ and $\underline{a} \times \underline{b}$

$$\underline{a} \cdot \underline{b} = \left(2\underline{i} - 5\underline{j} + \underline{k}\right).\left(4\underline{i} - 2\underline{j} - 9\underline{k}\right)$$

$$= (2)(4) + (-5)(-2) + (1)(-9)$$

$$= 9$$

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -5 & 1 \\ 4 & -2 & -9 \end{vmatrix}$$

$$= \begin{vmatrix} -5 & 1 \\ -2 & -9 \end{vmatrix} \underline{i} - \begin{vmatrix} 2 & 1 \\ 4 & -9 \end{vmatrix} \underline{j} + \begin{vmatrix} 2 & -5 \\ 4 & -2 \end{vmatrix} \underline{k}$$

$$= (45 + 2)\underline{i} - (-18 - 4)\underline{j} + (-4 + 20)\underline{k}$$

$$= 47\underline{i} + 22\underline{j} + 16\underline{k}$$

Question 3:
Find the $\|\underline{a}\times\underline{b}\|$, if $\underline{a}=\underline{i}-7\underline{j}+7\underline{k}$ and $\underline{b}=3\underline{i}-2\underline{j}+2\underline{k}$

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -7 & 7 \\ 3 & -2 & 2 \end{vmatrix} = \begin{vmatrix} -7 & 7 \\ -2 & 2 \end{vmatrix}\underline{i} - \begin{vmatrix} 1 & 7 \\ 3 & 2 \end{vmatrix}\underline{j} + \begin{vmatrix} 1 & -7 \\ 3 & -2 \end{vmatrix}\underline{k}$$

$$= (-14+14)\underline{i} - (2-21)\underline{j} + (-2+21)\underline{k}$$

$$= 0\underline{i} + 19\,\underline{j} + 19\underline{k}$$

Therefore,

$$\|\underline{a}\times\underline{b}\| = \sqrt{0^2+19^2+19^2}$$
$$= \sqrt{722}$$
$$= \sqrt{19}\sqrt{19}\sqrt{2} = 19\sqrt{2} \quad \text{OR} \quad 26.87$$

Question 4:
If $\underline{a} = 5\underline{i} - \underline{j} - 3\underline{k}$ and $\underline{b} = \underline{i} + 3\underline{j} - 5\underline{k}$. Find $\underline{a} + \underline{b}$ and $\underline{a} - \underline{b}$. Show that $\underline{a} + \underline{b}$ and $\underline{a} - \underline{b}$ are perpendicular.

$$\underline{a} + \underline{b} = (5+1)\underline{i} + (-1+3)\underline{j} + \left(-3 + (-5)\right)\underline{k}$$
$$\underline{a} + \underline{b} = 6\underline{i} + 2\underline{j} - 8\underline{k}$$

$$\underline{a} - \underline{b} = (5-1)\underline{i} + (-1-3)\underline{j} + (-3 - (-5))\underline{k}$$
$$= 4\underline{i} - 4\underline{j} + 2\underline{k}$$

Hence, the vectors are perpendicular if their scalar product is zero.

$$\left(\underline{a} + \underline{b}\right) \cdot (\underline{a} - \underline{b}) = 0$$
$$(6\underline{i} + 2\underline{j} - 8\underline{k}) \,.\, \left(4\underline{i} - 4\underline{j} + 2\underline{k}\right) = LHS$$
$$= (6)(4) + (2)(-4) + (-8)(2)$$
$$= (24) + (-8) + (-16)$$
$$=0 = RHS$$

Question 5:
Calculate the volume of parallelepiped , whose edges are represented by $\underline{a} = 2\underline{i} - 3\underline{j} + \underline{k}$, $\underline{b} = \underline{i} - \underline{j} + 2\underline{k}$ and $\underline{c} = 2\underline{i} + \underline{j} - \underline{k}$.

Volume of parallelepiped= $|\underline{a} \,.\, (\underline{b} \times \underline{c})|$

$$= \begin{vmatrix} 2 & -3 & 1 \\ 1 & -1 & 2 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= (2)\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} - (-3)\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} + (1)\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$$

$$= (2)(1-2) + (3)(-1-4) + (1+2)$$

$$= -2 \; - 15 + 3 = -14$$

Since Volume > 0. Thus,

Volume of parallelepiped=14 cubic units/ $units^3$.

Question 6:

Suppose that $\underline{a}$ and $\underline{b}$ are vectors with $\|\underline{a}\| = 2$, $\|\underline{b}\| = 5$ and $\|\underline{a} \times \underline{b}\| = 6$. Therefore, find $\|\underline{a} \cdot \underline{b}\|$.

Let θ be angle between $\underline{a}$ and $\underline{b}$. So, we have

$$\sin\theta = \frac{\|\underline{a} \times \underline{b}\|}{\|\underline{a}\|\|\underline{b}\|} = \frac{6}{2(5)}$$

$$= \frac{3}{5}$$

$$\theta = \sin^{-1}\left(\frac{3}{5}\right) = 36.87°$$

Given that

$$\cos\theta = \frac{\|\underline{a} \cdot \underline{b}\|}{\|\underline{a}\|\|\underline{b}\|} = \frac{\|\underline{a} \cdot \underline{b}\|}{2(5)} = \frac{\|\underline{a} \cdot \underline{b}\|}{10}$$

$$\cos 36.87° = \frac{4}{5}$$

$$\frac{\|\underline{a} \cdot \underline{b}\|}{10} = \frac{4}{5}$$

$$\|\underline{a} \cdot \underline{b}\| = 8$$

Question 7:
Find the value of α so that the three vectors $\underline{a} = \underline{i} - 2\underline{j} + 3\underline{k}$ and $\underline{b} = 4\underline{i} - \underline{k}$ and $\underline{c} = -8\underline{i} - 8\underline{j} + \alpha\underline{k}$ are coplanar

The vectors are coplanar if

$$\underline{a} \cdot (\underline{b} \times \underline{c}) = 0$$

$$\begin{vmatrix} 1 & -2 & 3 \\ 4 & 0 & -1 \\ -8 & -8 & \alpha \end{vmatrix} = (1)\begin{vmatrix} 0 & -1 \\ -8 & \alpha \end{vmatrix} - (-2)\begin{vmatrix} 4 & -1 \\ -8 & \alpha \end{vmatrix} + (3)\begin{vmatrix} 4 & 0 \\ -8 & -8 \end{vmatrix}$$

$$= (1)(0-8) - (-2)(4\alpha - 8) + (3)(-32-0)$$

$$= -8 + 8\alpha - 16 - 96$$

Therefore,

$$-8 + 8\alpha - 16 - 96 = 0$$
$$8\alpha - 120 = 0$$
$$\alpha = \frac{120}{8} = 15$$

IMPORTANT ANNOUNCEMENT

USM We lead

- Kindly send me an email via asyrafman@usm.my for any inquires or to set up an appointment for JIM319 consultation.

- Please visit e-learning portal religiously, if possible on weekly basis.

- Extra exercises will be uploaded from time to time.

- Do watch the pre-recorded e-kuliah or videos before attending WEBEX class for a better understanding.

Thank You