JIM 101: CALCULUS

Differentiation and Applications of Derivatives

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E-LECTURE

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WEBEX CLASS 4 AGENDA

JIM101 CALCULUS

05 DIFFERENTIATION

- ✓ Geometrical Meaning
- ✓ Definition
- ✓ Differentiation of Simple Algebraic Function
- ✓ Differentiation of Various Type of Functions

06 APPLICATIONS OF DERIVATIVES

- ✓ Tangent and Normal Lines
- ✓ Price Elasticity of Demand
- ✓ Maxima and Minima

Geometric Meaning of Differentiation

Differentiation is all about measuring change!
Measuring change in a linear function:

$$y = a + bx$$

a = intercept

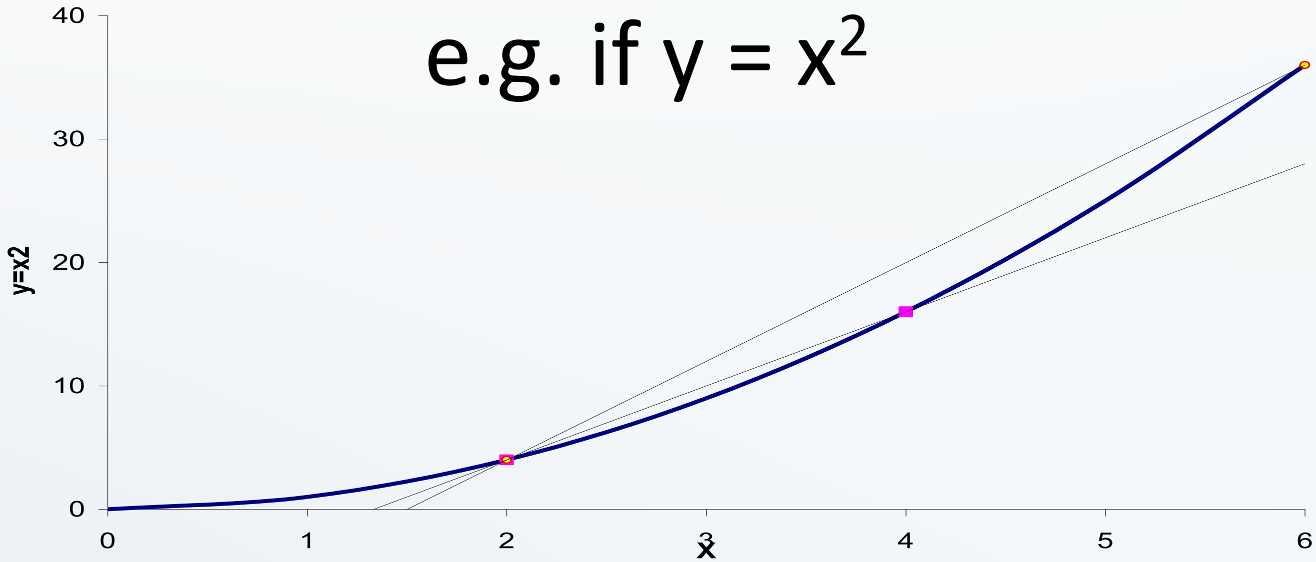
b = constant slope i.e. the impact of a unit change in x on the level of y

$$\mathbf{b} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Geometric Meaning of Differentiation

If the function is non-linear:

e.g. if $y = x^2$

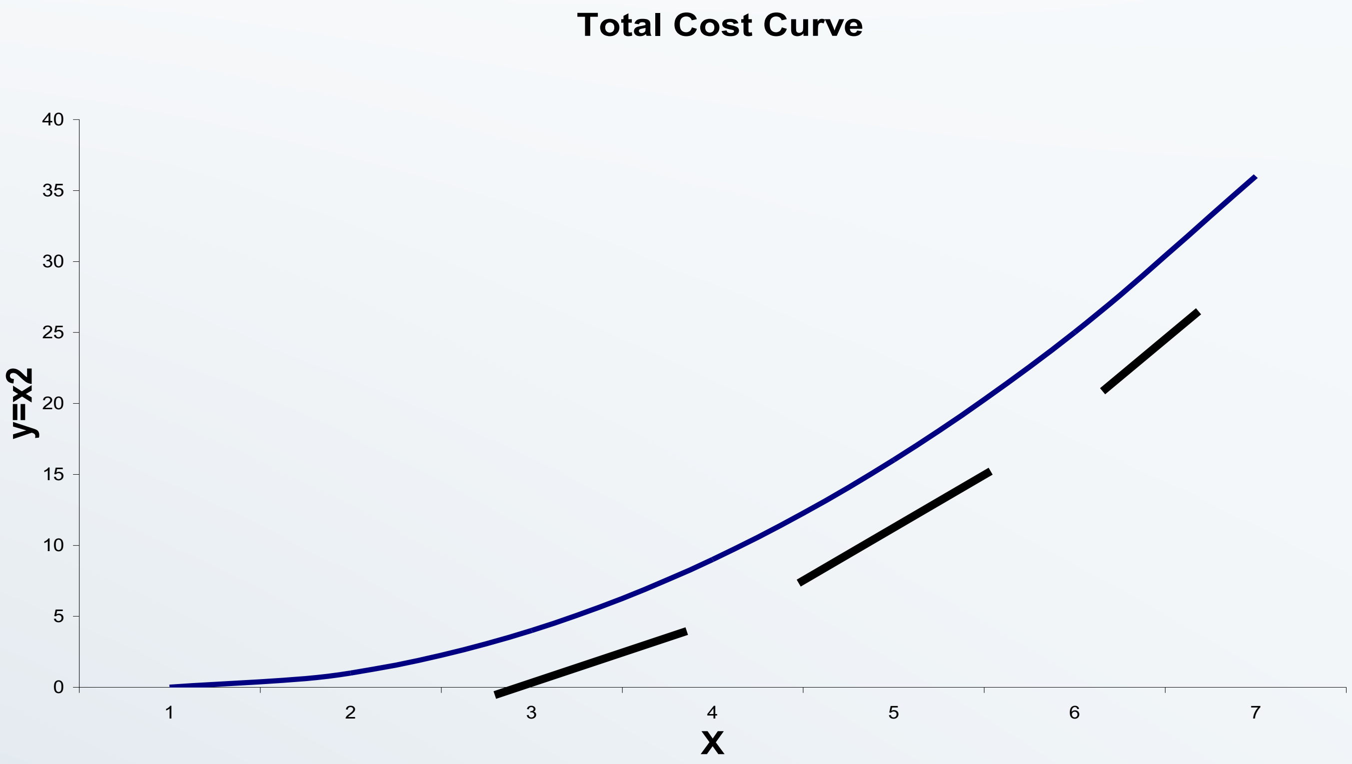


$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$ gives slope of the *line* connecting 2 points (x_1, y_1) and (x_2, y_2) on a curve

- (2,4) to (4,16): slope = $^{(16-4)}/_{(4-2)} = 6$
- (2,4) to (6,36): slope = $^{(36-4)}/_{(6-2)} = 8$

Geometric Meaning of Differentiation

The slope of a curve is equal to the slope of the line (or tangent) that touches the curve at that point



which is different for different values of x

Geometric Meaning of Differentiation

Example:A firms cost function is

$Y = X^2$

X	ΔX	Y	ΔY
0		0	
1	+1	1	+1
2	+1	4	+3
3	+1	9	+5
4	+1	16	+7

$$Y = X^2$$

$$Y+\Delta Y = (X+\Delta X)^2$$

$$Y+\Delta Y = X^2+2X.\Delta X+\Delta X^2$$

$$\Delta Y = X^2+2X.\Delta X+\Delta X^2 - Y$$

$$\text{since } Y = X^2 \Rightarrow \Delta Y = 2X.\Delta X+\Delta X^2$$

$$\frac{\Delta Y}{\Delta X} = 2X+\Delta X$$

The slope depends on X and ΔX

Geometric Meaning of Differentiation

The slope of the graph of a function is called the derivative of the function

$$f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

- The process of differentiation involves letting the change in x become arbitrarily small, i.e. letting $\Delta x \to 0$
- e.g if = $2X + \Delta X$ and $\Delta X \to 0$
- $\Rightarrow$ = $2X$ in the limit as $\Delta X \to 0$

Geometric Meaning of Differentiation

the slope of the non-linear function $Y = X^2$ is 2X

- **the slope tells us the change in y that results from a very small change in X**
- **We see the slope varies with X**

e.g. the curve at X = 2 has a slope = 4

and the curve at X = 4 has a slope = 8

- **In this example, the slope is steeper at higher values of X**

Definition of Derivative or Differentiation

DEFINITION The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

Question 1

By using the first principle, find the derivatives of $y = f(x) = 5x + 9$.

Solution

Differentiation of Simple Algebraic Function

Derivative of a Constant Function

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

Power Rule (General Version)

If n is any real number, then

$$\frac{d}{dx}x^n = nx^{n-1},$$

for all x where the powers x^n and x^{n-1} are defined.

Question 2

Differentiate the following functions with respect to x.

(a) $y = \sqrt{1300882525}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{1300882525}\right) = 0$$

(b) $y = -\frac{50}{51}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(-\frac{50}{51}\right) = 0$$

(c) $y = 7\pi$

$$\frac{dy}{dx} = \frac{d}{dx}(7\pi) = 0$$

(d) $y = 8.5\,t$, where t is a constant.

$$\frac{dy}{dx} = \frac{d}{dx}(8.5t) = 0$$

Question 3

Find $\frac{dy}{dx}$ of the following functions:

Solution

(a) $y = 4x^6$

$$\frac{dy}{dx} = \frac{d}{dx}(4x^6)$$
$$= 4(6)x^{6-1}$$
$$\frac{dy}{dx} = 24x^5$$

(b) $y = 4x^{-6}$

$$\frac{dy}{dx} = \frac{d}{dx}(4x^{-6})$$
$$= 4(-6)x^{-6-1}$$
$$\frac{dy}{dx} = -24x^{-7}$$

(c) $y = \sqrt[3]{x}$

$$\frac{dy}{dx} = \frac{d}{dx}(\sqrt[3]{x})$$
$$= \frac{d}{dx}\left(x^{\frac{1}{3}}\right)$$
$$= (\frac{1}{3})x^{\frac{1}{3}-1}$$
$$\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}}$$

(d) $y = -\frac{2}{3}x^2$

$$\frac{dy}{dx} = \frac{d}{dx}\left(-\frac{2}{3}x^2\right)$$
$$= -\frac{2}{3}(2)x^{2-1}$$
$$= -\frac{4}{3}x$$

Differentiation Rules

Derivative Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

Derivative Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

Question 4

Find $\frac{dy}{dx}$ of the following function, $y = 100x^{-22}$

Solution:

$$\frac{dy}{dx} = \frac{d}{dx}(100x^{-22})$$

$$= 100\frac{d}{dx}(x^{-22})$$

$$= 100(-22)x^{-22-1}$$

$$\frac{dy}{dx} = -2200\,x^{-23}$$

Question 5 (a)

Find $\frac{dy}{dx}$ of the following function:

(a) $y = 2x^5 + 4x^3 - 7x^2 + 6x + 9$

Solution:

$$\frac{dy}{dx} = \frac{d}{dx}(2x^5 + 4x^3 - 7x^2 + 6x + 9)$$
$$= \frac{d}{dx}(2x^5) + \frac{d}{dx}(4x^3) - \frac{d}{dx}(7x^2) + \frac{d}{dx}(6x) + \frac{d}{dx}(9)$$
$$= 2\frac{d}{dx}(x^5) + 4\frac{d}{dx}(x^3) - 7\frac{d}{dx}(x^2) + 6\frac{d}{dx}(x) + \frac{d}{dx}(9)$$
$$= 2(5x^4) + 4(3x^2) - 7(2x) + 6(1) + 0$$
$$= 10x^4 + 12x^2 - 14x + 6$$

Question 5 (b)

Find $\frac{dy}{dx}$ of the following function:

(b) $y = -\frac{2}{3}x^3 + \frac{5}{6}x$

Solution:

Differentiation Rules

Derivative Product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Question 6

Find $\frac{dy}{dx}$, if $y = 5x^6(7x^2 + 4x)$.

Solution:

Differentiation Rules

Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

Question 7

Find $\frac{dy}{dx}$, if $y = \frac{4x+3}{2x-1}, x \neq \frac{1}{2}$

Solution:

Higher Order Differentiation

Higher Order Derivatives

$y = f(x)$	Original Function (position /distance/ height)
$y' = f'(x) = \frac{dy}{dx}$	First Derivative (velocity)
$y'' = f''(x) = \frac{d^2y}{dx^2}$	Second Derivative (acceleration)
$y''' = f'''(x) = \frac{d^3y}{dx^3}$	Third Derivative (jerk)
$y^4 = f^{IV}(x) = \frac{d^4y}{dx^4}$	Fourth Derivative

Question 8

Find the second order derivative of $y = x^3 + 1$.

Solution:

The Chain Rule

THEOREM 2—The Chain Rule If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

Question 9

Find the derivative, $\frac{dy}{dx}$ for $y = (5x - 3)^7$.

Solution

Differentiation of Trigonometric Functions

The derivative of the sine function is the cosine function:

$$\frac{d}{dx}(\sin x) = \cos x.$$

The derivative of the cosine function is the negative of the sine function:

$$\frac{d}{dx}(\cos x) = -\sin x$$

The derivatives of the other trigonometric functions:

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

Question 10

Find $\frac{dy}{dx}$, if $y = \frac{\sin 3x}{\cos 4x}$,

Solution:

Differentiation of Logarithmic Functions

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

$$\frac{d}{dx}\ln[f(x)] = \frac{1}{f(x)} \bullet f'(x)$$

$$\frac{d}{dx}\log_a(x) = \frac{1}{x \bullet \ln(a)}$$

$$\frac{d}{dx}\log_a[f(x)] = \frac{1}{f(x) \bullet \ln(a)} \bullet f'(x)$$

Question 11

Find the derivative, $\frac{dy}{dx}$ for $y = \ln(x^3 + 9)$.

Solution

Let $u = x^3 + 9$, then we have $y = \ln u$

Differentiate with respect to x and u respectively:

$$\frac{du}{dx} = \frac{d}{dx}(x^3 + 9) \qquad \frac{dy}{du} = \frac{d}{du}(\ln u)$$

$$\frac{du}{dx} = 3x^2 \qquad \frac{dy}{du} = \frac{1}{u}$$

By using Chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \left(\frac{1}{u}\right)(3x^2)$$

$$= \frac{3x^2}{x^3 + 9}$$

Differentiation of Exponential Functions

Derivatives of Exponential Functions

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}e^u = e^u \frac{du}{dx}$$

$$\frac{d}{dx}a^x = a^x \ln a$$

$$\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}$$

Question 12

(a) Find the derivative, $\frac{dy}{dx}$ for $y = 2x^5 - 3e^{6x}$

Solution

$$\frac{dy}{dx} = \frac{d}{dx}(2x^5 - 3e^{6x})$$

$$= 2\frac{d}{dx}(x^5) - 3\frac{d}{dx}(e^{6x})$$

$$= 10x^4 - 3(e^{6x})\frac{d}{dx}(6x)$$

$$= 10x^4 - 18(e^{6x})$$

(b) Find the derivative, $\frac{dy}{dx}$ for $y = (\frac{11}{17})^x$

Solution

$$\frac{dy}{dx} = \frac{d}{dx}\left[\left(\frac{11}{17}\right)^x\right]$$

$$\frac{dy}{dx} = \left(\frac{11}{17}\right)^x \ln\frac{11}{17}$$

Differentiation of Implicit Functions

Implicit Differentiation

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with dy/dx on one side of the equation and solve for dy/dx.

Question 13

Given that $x^2y^2 + xy = 2$, find $\frac{dy}{dx}$ by using implicit differentiation.

Solution

Differentiate both side with respect to x.

$$\frac{d}{dx}(x^2y^2 + xy) = \frac{d}{dx}(2)$$

$$\frac{d}{dx}(x^2y^2) + \frac{d}{dx}(xy) = \frac{d}{dx}(2)$$

$$x^2\frac{d}{dx}(y^2) + y^2\frac{d}{dx}(x^2) + x\frac{d}{dx}(y) + y\frac{d}{dx}(x) = 0$$

$$x^2\frac{d}{dy}(y^2)\frac{dy}{dx} + y^2\frac{d}{dx}(x^2) + x\frac{d}{dy}(y)\frac{dy}{dx} + y\frac{d}{dx}(x) = 0$$

$$x^2(2y)\frac{dy}{dx} + 2xy^2 + x\frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx}(2x^2\mathrm{y} + \mathrm{x}) = -y - 2xy^2$$

$$\frac{dy}{dx} = \frac{-y - 2xy^2}{x + 2xy^2} = \frac{-y(1 + 2xy^2)}{x(1 + 2xy^2)}$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

Differentiation of Parametric Functions

A parametrized curve

$$x = f(t), \qquad y = g(t)$$

is differentiable at t if f and g are differentiable at t. At a point on an differentiable parametrized curve where y is also a differentiable function of x, the derivative $\frac{dy}{dt}$, $\frac{dx}{dt}$, and $\frac{dy}{dx}$ are related by Chain Rule:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \qquad \text{or} \qquad \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{1}{\frac{dx}{dt}}$$

if $\frac{dx}{dt} \neq 0$.

Question 14

Find $\frac{dy}{dx}$ in terms of t for the following parametric equations:

$$x = t^3 - t, \qquad y = 4 - t^2$$

Solution

Differentiate both with respect to t

$$\frac{dx}{dt} = \frac{d}{dt}(t^3 - t) = 3t^2 - 1$$

$$\frac{dy}{dt} = \frac{d}{dt}(4 - t^2) = -2t$$

By using theorem of differentiation of parametric functions:

$$\frac{dy}{dx} = \frac{dy}{dt} . \frac{dt}{dx}$$

$$= (-2t)(\frac{1}{3t^2 - 1})$$

$$\frac{dy}{dx} = -\frac{2t}{3t^2 - 1}$$

Differentiation of Inverse Trigonometric Functions

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}\left(\sin^{-1}x\right)=\frac{1}{\sqrt{1-x^2}}, x\neq\pm1$$

$$\frac{d}{dx}\left(\cos^{-1}x\right)=\frac{-1}{\sqrt{1-x^2}}, x\neq\pm1$$

$$\frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\cot^{-1}x\right)=\frac{-1}{1+x^2}$$

$$\frac{d}{dx}\left(\sec^{-1}x\right)=\frac{1}{|x|\sqrt{x^2-1}}, x\neq\pm1,0$$

$$\frac{d}{dx}\left(\csc^{-1}x\right)=\frac{-1}{|x|\sqrt{x^2-1}}, x\neq\pm1,0$$

Question 15

Find the derivative, $\frac{dy}{dx}$ for $y = \cos^{-1} 5x$.

Solution

Let $u = 5x$, then we have $y = \cos^{-1} u$

Differentiate with respect to x and u respectively:

$$\frac{du}{dx} = \frac{d}{dx}(5x) \qquad \frac{dy}{du} = \frac{d}{du}(\cos^{-1} u)$$

$$\frac{du}{dx} = 5 \qquad \frac{dy}{du} = -\frac{1}{\sqrt{1-u^2}}$$

By using Chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$$

$$= \left(-\frac{1}{\sqrt{1-u^2}}\right)(5)$$

$$= -\frac{5}{\sqrt{1-(5x)^2}} = -\frac{5}{\sqrt{1-25x^2}}$$

Differentiation of Hyperbolic Functions

$$\frac{d}{dx}(\sinh ax) = a\cosh ax$$

$$\frac{d}{dx}(\cosh ax) = a\sinh ax$$

$$\frac{d}{dx}(\tanh ax) = a\,\mathrm{sech}^2\, ax$$

$$\frac{d}{dx}(\mathrm{csch}\, ax) = -a\,\mathrm{csch}\, ax \coth ax$$

$$\frac{d}{dx}(\mathrm{sech}\, ax) = -a\,\mathrm{sech}\, ax \tanh ax$$

$$\frac{d}{dx}(\coth ax) = -a\,\mathrm{csch}^2\, ax$$

Question 16

Find the derivative, $\frac{dy}{dx}$ for $y = sinh\,(5x^4 - 9)$.

Solution

Let $u = 5x^4 - 9$, then we have $y = \sinh u$

Differentiate with respect to x and u respectively:

$$\frac{du}{dx} = \frac{d}{dx}(5x^4 - 9) \qquad \frac{dy}{du} = \frac{d}{du}(\sinh u)$$

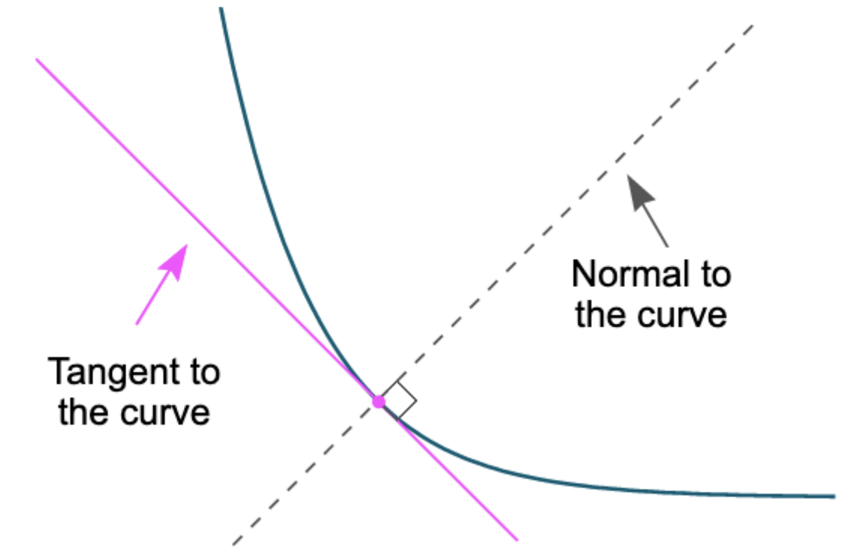
$$\frac{du}{dx} = 20x^3 \qquad \frac{dy}{du} = \cosh u$$

By using Chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= (\cosh u)(20x^3)$$
$$= 20x^3 \cosh u$$
$$= 20x^3 \cosh(5x^4 - 9)$$

Application of Derivatives I: Tangent and Normal Lines

- ✓ We often need to find tangents and normals to curves when we are analysing forces acting on a moving body.
- ✓ A **tangent** to a curve is a line that touches the curve at one point and has the same **slope** as the curve at that point.
- ✓ A **normal** to a curve is a line **perpendicular** to a tangent to the curve.



Graph showing the tangent and the normal to a curve at a point.

Applications II

- how does demand change with a change in price……

- e_d=

$$\frac{proportional\ change\ in\ demand}{proportional\ change\ in\ price}$$

$$= \frac{\Delta Q}{Q} \Big/ \frac{\Delta P}{P} = \frac{\Delta Q}{\Delta P} \cdot \frac{P}{Q}$$

Point elasticity of demand

$$e_d = \frac{dQ}{dP} \cdot \frac{P}{Q}$$

e_d is negative for a downward sloping demand curve

–Inelastic demand if $| e_d | < 1$

–Unit elastic demand if $| e_d | = 1$

–Elastic demand if $| e_d | > 1$

Example 2

If the (inverse) Demand equation is

$P = 200 - 40\ln(Q+1)$

Calculate the price elasticity of demand when $Q = 20$

- Price elasticity of demand: $e_d = \frac{dQ}{dP} \cdot \frac{P}{Q} < 0$
- P is expressed in terms of Q,

$$\frac{dP}{dQ} = -\frac{40}{Q+1}$$

- Inverse rule $\Rightarrow \frac{dQ}{dP} = -\frac{Q+1}{40}$
- Hence, $e_d = -\frac{Q+1}{40} \cdot \frac{P}{Q} < 0$
- Q is 20 $\Rightarrow e_d = -\frac{21}{40} \cdot \frac{78.22}{20} = -2.05$

(where $P = 200 - 40\ln(20+1) = 78.22$)

Application of Derivatives III: Maxima and Minima

Consider the function $g(x) = 3x^4 + 16x^3 + 24x^2 + 3$.

(i). Find all the critical points of $g(x)$.

(ii). Classify each of the critical point you obtained in part (i) as a maximum or a minimum (or neither) by using the first derivative test.

Solution (i)

Application of Derivatives III: Maxima and Minima

Consider the function $g(x)=3x^4+16x^3+24x^2+3$.

(i). Find all the critical points of $g(x)$.

(ii). Classify each of the critical point you obtained in part (i) as a maximum or a minimum (or neither) by using the first derivative test.

Solution (ii)

Extra Exercise (Please Attempt)

1. By using the first principle, find the derivatives of $y = f(x) = \frac{4x}{x+1}$.

2. Find $\frac{dy}{dx}$, if $y = -\frac{3}{4}ax^{\frac{2}{5}} + 5bx + 4c$ where a, b, c are constant.

3. Find the derivative of $y = (6x^2 + 7)^{-1}$ by using the chain rule.

4. Find $\frac{dy}{dx}$, if $y = \frac{\tan x}{x}$.

5. Find $\frac{dy}{dx}$, if $y = e^x + (x^4 + 1)\ln x + 5$.

6. Compute the derivative of $f(x) = (2x^4 - 3x + 5)(x^2 - \sqrt{x} + \frac{2}{x})$.

Extra Exercise (Please Attempt)

7. Given that $x^2y^2 - 2x = 4 - 4y$, find $\frac{dy}{dx}$ by using implicit differentiation.

8. Compute the derivative, $\frac{dy}{dx}$ for $y = \tan^{-1}(x^3)$.

9. Consider the function $f(x) = x^4 - 2x^2$.
(a) Find all the critical points of $f(x)$.
(b) Classify each of the critical point you obtained in (a) as minimum or maximum (or neither) by using first derivative test.

ANNOUNCEMENT

- Kindly send me an email via asyrafman@usm.my for any inquires or to set up an appointment for JIM101 consultation.
- The tentative due date for e-assignment 1 is on **29th January 2020**.
- The intensive course will be held on 28th January 2020 until 13th February 2020. More info will be announced via e-portal from time to time.

WHAT TO FOCUS FOR PB ???

The continuous assessment will cover the topics from Webex 1 until Webex 4. Focus should be given on:

- Equality of complex number
- Polar form of complex number
- Argand diagram
- Finding range and domain for inverse function
- Solving the limit via properties of limits
- Continuous function
- Continuity test
- Product Rule
- Quotient Rule
- Differentiation of trigonometric functions.

Kami Memimpin *We Lead*

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THANK YOU