Algebraic Operations with Fuzzy Numbers

Definition 1. A fuzzy number $\widetilde{M}$ is a convex normalized fuzzy set $\widetilde{M}$ of the real line $\mathbb{R}$ such that

(1) It exists exactly one $x_0 \in \mathbb{R}$ with $\mu_{\widetilde{M}}(x_0) = 1$ (x_0 is called the mean value of M).

(2) $\mu_{\widetilde{M}}(x)$ is piecewise continuous.

Definition 2. A fuzzy number $\widetilde{M}$ is called positive (negative) if its membership function is such that $\mu_{\widetilde{M}}(x) = 0, \forall x < 0 (\forall x > 0)$.

Example 1. The following fuzzy sets are fuzzy numbers:

$$\text{approximately } 5 = \{(3,.2),(4,.6),(5,1),(6,.7),(7,.1)\}$$
$$\text{approximately } 10 = \{(8,.3),(9,.7),(10,1),(11,.7),(12,.3)\}$$

But $\{(3,.8),(4,1),(5,I),(6,.7)\}$ is not a fuzzy number because $\mu(4)$ and also $\mu(5) = 1$

Definition 3. A binary operation * in $\mathbb{R}$ is called increasing (decreasing) if

$$\text{for } x_1 > y_1 \quad \text{and} \quad x_2 > y_2,$$
$$x_1 * x_2 > y_1 * y_2 (x_1 * x_2 < y_1 * y_2)$$

Example 2.

$$f(x,y) = x + y \quad \text{is an increasing operation.}$$
$$f(x,y) = x \cdot y \quad \text{is an increasing operation on } \mathbb{R}+.$$
$$f(x,y) = -(x+y) \quad \text{is a decreasing operation.}$$

If the normal algebraic operations $+, -, ., :$ are extended to operations on fuzzy numbers, they shall be denoted by $\oplus, \ominus, \odot,$.

Theorem 1. *If $\widetilde{M}$ and $\widetilde{N}$ are fuzzy numbers whose membership functions are continuous and surjective from $\mathbb{R}$ to $[0,1]$ and $*$ is a continuous increasing (decreasing) binary operation, then $\widetilde{M} \circledast \widetilde{N}$ is a fuzzy number whose membership function is continuous and surjective from $\mathbb{R}$ to $[0,1]$.*

Dubois and Prade present procedures to determine the membership functions $\mu_{\widetilde{M} \odot \widetilde{N}}$ on the basis of $\mu_{\widetilde{M}}$ and $\mu_{\widetilde{N}}$.

Theorem 2. *If $\widetilde{M}, \widetilde{N} \in F(\mathbb{R})$ with $\mu_{\widetilde{N}}(x)$ and $\mu_{\widetilde{M}}(x)$ continuous membership functions, then by application of the extension principle for the binary operation $* : \mathbb{R} \otimes \mathbb{R} \to \mathbb{R}$ the membership function of the fuzzy number $\widetilde{M} \circledast \widetilde{N}$ is given by*

$$\mu_{\widetilde{M} \circledast \widetilde{N}}(z) = \sup_{z = x * y} \min\{\mu_{\widetilde{M}}(x), \mu_{\widetilde{N}}(y)\}$$

Properties of the extended operation $\circledast$

Remark 1.

1.*For any commutative operation $*$, the extended operation $\circledast$ is also commutative.*

2.*For any associative operation $*$, the extended operation $\circledast$ is also associative.*

Special Extended Operations

For unary operations $f : X \to Y, X = X_1$ (see definitions 5-1), the extension principle reduces for all $\widetilde{M} \in F(\mathbb{R})$ to

$$\mu_{f(\widetilde{M})}(z) = \sup_{x \in f^{-1}(z)} \mu_{\widetilde{M}}(x)$$

Example 3. Consider the following examples.

(1) For$f(x) = -x$, the opposite of a fuzzy number $\widetilde{M}$ is given by $-\widetilde{M} = \{(x, \mu_{-\widetilde{M}}(x)) | x \in X\}$, where $\mu_{-\widetilde{M}}(x) = \mu_{\widetilde{M}}(-x)$.

(2) If $f(x) = \frac{1}{x}$, then the inverse of a fuzzy number $\widetilde{M}$ is given by $\widetilde{M}^{-1} = \{(x, \mu^{-1}_{\widetilde{M}}(x)) | x \in X\}$, where $\mu^{-1}_{\widetilde{M}} = \mu_{\widetilde{M}}(\frac{1}{x})$.

(3) For $\lambda \in \mathbb{R} \setminus \{0\}$ and $f(x) = \lambda \cdot x$, then the scalar multiplication of a fuzzynumber is given by $\lambda M = \{(x, \mu_{\lambda \widetilde{M}}(x)) | x \in X\}$, where $\mu_{\lambda \widetilde{M}}(x) = \mu_{\widetilde{M}}(\lambda \cdot x)$.

In the following, we shall apply the extension principle to binary operations. A generalization to n-ary operations is straightforward.

Extended Addition. Since addition is an increasing operation according to **Theorem 1**, we get for the extended addition $\oplus$ of fuzzy numbers that $f(\widetilde{N}, \widetilde{M}) = \widetilde{N} \oplus \widetilde{M}, \widetilde{N}, \widetilde{M} \in F(\mathbb{R})$ is a fuzzy number-that is, $\widetilde{N} \oplus \widetilde{M} \in F(\mathbb{R})$.

The properties of $\oplus$ are as follows:

(1) $\ominus(\widetilde{M} \oplus \widetilde{N}) = (\ominus\widetilde{M}) \oplus (\ominus\widetilde{N})$.
(2) $\oplus$ is commutative.
(3) $\oplus$ is associative.
(4) $0 \in \mathbb{R} \subseteq F(\mathbb{R})$ is the neutral element for $\oplus$, that is,$\widetilde{M} \oplus 0 = \widetilde{M}, \forall \widetilde{M} \in F(\mathbb{R})$.
(5) For $\oplus$ there does not exist an inverse element, that is, $\forall \widetilde{M} \in F(\mathbb{R}) \setminus \mathbb{R} : \widetilde{M} \oplus (\ominus\widetilde{M}) \neq 0 \in \mathbb{R}$.

Extended Product. Multiplication is an increasing operation on $\mathbb{R}^+$ and a decreasing operation on $\mathbb{R}^-$. Hence, according to theorem 5-1, the product of positive fuzzy numbers or of negative fuzzy numbers results in a positive fuzzy number. Let $\widetilde{M}$ be a positive and $\widetilde{N}$ a negative fuzzy number. Then $\ominus\widetilde{M}$ is also negative and $\widetilde{M} \odot \widetilde{N} = \ominus(\ominus\widetilde{M} \odot \widetilde{N})$ results in a negative fuzzy number.

The properties of $\odot$ are as follows:

(1) $(\ominus\widetilde{M}) \odot \widetilde{N} = \ominus(\widetilde{M} \odot \widetilde{N})$.
(2) $\odot$ is commutative.
(3) $\odot$ is associative.
(4) $\widetilde{M} \odot 1 = \widetilde{M}, 1 \in \mathbb{R} \subseteq F(\mathbb{R})$ is the neutral element for $\odot$, that is, $\widetilde{M} \odot 1 = \widetilde{M}, \forall \widetilde{M} \in F(\mathbb{R})$.
(5) For $\odot$ there does not exist an inverse element, that is, $\forall \widetilde{M} \in F(\mathbb{R}) \setminus \mathbb{R} : \widetilde{M} \odot \widetilde{M}^{-1} \neq 1$.

Theorem 3. *If $\widetilde{M}$ is either a positive or a negative fuzzy number and $\widetilde{N}$ and $\widetilde{P}$ are both either positive or negative fuzzy numbers, then*

$$\widetilde{M} \odot (\widetilde{N} \oplus \widetilde{P}) = (\widetilde{M} \odot \widetilde{N}) \oplus (\widetilde{M} \odot \widetilde{P})$$

Extended Subtraction. Subtraction is neither an increasing nor a decreasing operation. Therefore theorem 5-1 is not immediately applicable. The operation $\widetilde{M} \ominus \widetilde{N}$ can, however, always be written as $\widetilde{M} \ominus \widetilde{N} = \widetilde{M} \oplus (\ominus\widetilde{N})$.

Applying the extension principle [Dubois and Prade 1979] yields

$$\begin{aligned} \mu_{\widetilde{M} \ominus \widetilde{N}}(z) &= \sup_{z=x-y} \min(\mu_{\widetilde{M}}(x), \mu_{\widetilde{N}}(y) \\ &= \sup_{z=x+y} \min(\mu_{\widetilde{M}}(x), \mu_{\widetilde{N}}(-y)) \\ &= \sup_{z=x+y} \min(\mu_{\widetilde{M}}(x), \mu_{-\widetilde{N}}(y) \end{aligned}$$

Thus $\widetilde{M} \ominus \widetilde{N}$ is a fuzzy number whenever $\widetilde{M}$ and $\widetilde{N}$ are.

Extended Division. Division is also neither an increasing nor a decreasing operation. If $\widetilde{M}$ and $\widetilde{N}$ are strictly positive fuzzy numbers, however (that is, $\mu_{\widetilde{M}}(x) = 0$ and $\mu_{\widetilde{N}}(x) = 0 \forall x \leq 0$), we obtain in analogy to the extended subtraction

$$\begin{aligned} \mu_{\widetilde{M} \ominus \widetilde{N}}(z) &= \sup_{z=x/y} \min(\mu_{\widetilde{M}}(x), \mu_{\widetilde{N}}(y)) \\ &= \sup_{z=xy} \min\left(\mu_{\widetilde{M}}(x), \mu_{\widetilde{N}}\left(\frac{1}{y}\right)\right) \\ &= \sup_{z=xy} \min(\mu_{\widetilde{M}}(x), \mu_{\widetilde{N}}^{-1}(y)) \end{aligned}$$

$\widetilde{N}^1$ is a positive fuzzy number. Hence theorem 5-1 can now be applied. The same is true if $\widetilde{M}$ and $\widetilde{N}$ are both strictly negative fuzzy numbers.

Example 4. Let $\widetilde{M} = \{(I, .3), (2, 1), (3, .4)\}$, $\widetilde{N} = \{(2, .7), (3, I), (4, .2)\}$

Then, $\widetilde{M} \odot \widetilde{N} = \{(2, .3), (3, .3), (4, .7), (6, 1), (8, .2), (9, .4), (12, .2)\}$

Definition 4. A fuzzy number $\widetilde{M}$ is of *LR-type* if there exist reference functions L (for left), R (for right), and scalars $\alpha > 0, \beta > 0$with

$$\mu_{\widetilde{M}}(x) = \begin{cases} L\left(\dfrac{m-x}{\alpha}\right) & \text{for} \quad x \leq m \\ R\left(\dfrac{x-m}{\beta}\right) & \text{for} \quad x \geq m \end{cases}$$

m, called the mean value of $\widetilde{M}$ it, is a real number, and α and β are called the left and right spreads, respectively. Symbolically,$\widetilde{M}$ it is denoted by $(m, \alpha, \beta)_{LR}$ (Look at the internet how does the L-R fuzzy number looks like!)

For *L(z)*, different functions can be chosen. Dubois and Prade [1988a, p. 50] mention, for instance, $L(x) = \max(0, 1-x)^p, L(x) = \max(0, 1-x^p)$, with $p > 0$ and $L(x) = e^{-x}$ or $L(x) = e^{-x^2}$. These examples already give an impression of the wide scope of *L(z)*. One problem, of course, is to find the appropriate function in a specific context.

Example 5. Let

$$L(x) = \frac{1}{1+x^2}$$
$$R(x) = \frac{1}{2|x|}$$
$$\alpha = 2, \beta = 3, / = 5$$

Then

$$\mu_{\widetilde{M}}(x) = \begin{cases} L\left(\dfrac{5-x}{2}\right) = \dfrac{1}{1+\left(\dfrac{5-x}{2}\right)} & \text{for} \quad x \leq 5 \\ R\left(\dfrac{x-5}{3}\right) = \dfrac{1}{1+\left|\dfrac{2(x-5)}{3}\right|} & \text{for} \quad x \geq 5 \end{cases}$$

If m is not a real number but an interval $[\underline{m}, \overline{m}]$, then the fuzzy set $\widetilde{M}$ is not a fuzzy number but a fuzzy interval. Accordingly, a fuzzy interval in LR representation can be defined as follows:

Definition 5. A *fuzzy interval* $\widetilde{M}$ if is of LR-type if there exist shape functions L and R and four parameters $(\underline{m}, \overline{m}) \in \mathbb{R}^2 \cup \{-\infty, +\infty, \alpha, \beta\}$ and the membership function of $\widetilde{M}$ is

$$\mu_{\widetilde{M}}(x) = \begin{cases} L\left(\dfrac{\underline{m}-x}{\alpha}\right) & \text{for} \quad x \leq \underline{m} \\ 1 & \text{for} \quad \underline{m} \leq x \leq \overline{m} \\ R\left(\dfrac{x-\overline{m}}{\beta}\right) & \text{for} \quad x \geq \underline{m} \end{cases}$$

The fuzzy interval is then denoted by

$$\widetilde{M} = (\underline{m}, \overline{m}, \alpha, \beta)_{LR}$$

This definition is very general and allows quantification of quite different types of information ; for instance, if $\widetilde{M}$ is supposed to be a real crisp number for $m \in \mathbb{R}$,

$$\widetilde{M}(m, m, 0, 0)_{LR}, \forall L, \forall R$$

If $\widetilde{M}$ is a crisp interval,

$$\widetilde{M} = (a, b, 0, 0)_{LR}, \forall L, \forall R$$

and if $\widetilde{M}$ is a "trapezoidal fuzzy number", $L(x) = R(x) = \max(0, 1 - x)$ is implied.

Theorem 4. *Let $\widetilde{M}, \widetilde{N}$ be two fuzzy numbers of LR-type:*

$$\widetilde{M} = (m, \alpha, \beta)_{LR}, \quad \widetilde{N} = (n, \gamma, \delta)_{LR}$$

Then,

(1) $(m, \alpha, \beta)_{LR} \oplus (n, \gamma, \delta)_{LR} = (m + n, \alpha + \gamma, \beta + \delta)_{LR}.$

(2) $-(m, \alpha, \beta)_{LR} = (-m, \beta, \alpha)_{LR}.$

(3) $(m, \alpha, \beta)_{LR} \ominus (n, \gamma, \delta)_{LR} = (m - n, \alpha + \gamma, \beta + \delta)_{LR}$

Example 6.

$$L(x) = R(x) = \frac{1}{1 + x^2}$$
$$\widetilde{M} = (1, .5, .8)_{LR}$$
$$\widetilde{N} = (2, .6, .2)_{LR}$$
$$\widetilde{M} \oplus \widetilde{N} = (3, 1.1, 1)_{LR}$$
$$\widetilde{O} = (2, .6, .2)_{LR}$$
$$\ominus \widetilde{O} = (-2, .2, .6)^{LR}$$
$$\widetilde{M} \ominus \widetilde{O} = (-1, .7, 1.4)_{LR}$$

Theorem 5. *Let, $\widetilde{M}, \widetilde{N}$ be fuzzy numbers as in definition 5-3; then*

$$(m, \alpha, \beta)_{LR} \odot (n, \gamma, \delta)_{LR} \approx (mn, m\gamma + n\alpha, m\delta + n\beta)_{LR}$$

for $\widetilde{M}, \widetilde{N}$ positive;

$$(m, \alpha, \beta)_{LR} \odot (n, \gamma, \delta)_{LR} \approx (mn, n\alpha - m\delta, n\beta - m\gamma)_{LR}$$

for $\widetilde{N}$ positive, $\widetilde{M}$ negative, and

$$(m, \alpha, \beta)_{LR} \odot (n, \gamma, \delta)_{LR} = (mn, -n\beta - m\delta, n\alpha - m\gamma)_{LR}$$

for $\widetilde{M}, \widetilde{N}$ negative.

The following example shows an application of the theorem.

Example 7. Let $\widetilde{M} = (2, .2, .1)_{LR}$ and $\widetilde{N}(3, .1, .3)_{LR}$ be fuzzy numbers of LR-type with reference functions

$$L(z) = R(z) = \begin{cases} 1 & -1 \le z \le 1 \\ 0 & \text{else} \end{cases}$$

If we are interested in the LR-representation of $\widetilde{M} \odot \widetilde{N}$, we prove the conditions of the previous theorem and apply it. Thus, with

$$\begin{aligned} \mu_{\widetilde{M}}(x) &= \begin{cases} L\left(\dfrac{2-x}{.2}\right) & x \le 2 \\ R\left(\dfrac{x-2}{.1}\right) & x \ge 2 \end{cases} \\ &= \begin{cases} 1 & -1 \le \dfrac{2-x}{.2} \le 1 \quad \text{and} \quad -1 \le \dfrac{x-2}{.1} \le 1 \\ 0 & \text{else} \end{cases} \\ &= \begin{cases} 1 & 1.9 \le x \le 2.1 \\ 0 & \text{else} \end{cases} \end{aligned}$$

it follows that $\widetilde{M}$ is positive.

$$\begin{aligned} \mu_{\widetilde{N}}(x) &= \begin{cases} L\left(\dfrac{3-x}{.1}\right) & x \le 3 \\ R\left(\dfrac{x-3}{.3}\right) & x \ge 3 \end{cases} \\ &= \begin{cases} 1 & 2.9 \le x \le 3.1 \\ 0 & \text{else} \end{cases} \end{aligned}$$

shows that $\widetilde{N}$ is positive.

Following the theorem for the case in which $\widetilde{M}$ and $\widetilde{N}$ are positive, we obtain

$$\widetilde{M} \odot \widetilde{N} \approx (2 \cdot 3, 2 \cdot 0.1 + 3 \cdot 0.2, 2 \cdot 0.3 + 3 \cdot 0.1)_{LR} == (6, .8, .9)_{LR}$$