SCHOOL OF MATHEMATICAL SCIENCES

ACADEMIC SESSION 2019/2020 - SEMESTER I

MSS414 – TOPICS IN PURE MATHEMATICS

Assignment 2:

Solution of Fuzzy Heat Equation

LECTURER:

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1. INTRODUCTION

A partial differential equation (PDE) is an equation that relates a function of more than one variable to its partial derivatives. PDEs are more ideal than ordinary differential equation (ODE) when dealing with functions of several variables which can be solved by hand or can be used to create a computer model. When dealing with real-life problems like modelling the temperature in a thin metal bar, we have to deal with two variables simultaneously which are position and time. Several methods for solving PDEs have been proposed to solve this modeling problems. In this project, we are using separation of variable to solve the heat equation.

However, PDEs are not always the best option when dealing with real-life problems. This is because the model is not always accurate due to incomplete knowledge and information of the modeling system. One of them is the initial value assigned to the model. To handle uncertainty quantities, researchers proposed several new concepts includes fuzzy set theory. This theory is able to deal with differential equation possessing uncertainties at initial values. Hence, we solve the heat equation by using fuzzy partial differential equation (FPDE) and fuzzy set theory.

Fuzzy sets were introduced in 1965 by Zadeh, where the author emphasized that a number can be classified into certain membership function rather than we represent it as a discrete or crisp number. Study of fuzzy partial differential equations (FPDEs) means the generalization of partial differential equations (PDEs) in fuzzy sense. While doing modelling of real situation in terms of partial differential equation, we see that the variables and parameters involve in the equations are uncertain. We express this impreciseness and uncertainties in terms of fuzzy numbers. So we come across with fuzzy partial differential equation. To study the solution of fuzzy initial and boundary value problems, we need the concept of differentiability of fuzzy-valued function. Hence, we decided to use Seikkala differentiability of a fuzzy-valued function.

2. OBJECTIVES

For this project, we should be able to achieve the following objectives:

- To solve heat equation and fuzzy heat equation.
- To study the effect of fuzziness on heat equation.
- To investigate the relationship between partial differential equation and fuzzy partial differential equation.
- Able to understand the application of fuzzy set theory in real world.

3. METHODOLOGY

In this section, we explain the modeling of heat equation and solution for partial differential equation (PDE) and fuzzy partial differential equation (FPDE).

a) **Partial differential equation (PDE)**

Solving the heat equation using method of separation of variables:

The heat equation is in the form

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

Initial condition: $u(x,0) = f(x)$
Boundary condition: $u(0,t) = 0 \quad u(L,t) = 0$

By using method of separation of variables, we will get the solution to the partial differential equation

$$u(x,t) = \sum_{n=1}^{M} B_n \sin(\frac{n\pi x}{L}) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Take the limit as M→∞ and the solution becomes

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi x}{L}) e^{-k\left(\frac{n\pi}{L}\right)^2 t} \qquad (1)$$

The solution will satisfy any initial condition that can be written in the form

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi x}{L})$$

We can determine the B_n when we find the Fourier sine series of initial condition.

$$B_n = \frac{2}{L}\int_0^L f(x) sin\left(\frac{n\pi x}{L}\right) dx \qquad n = 1,2,3, \ldots$$

$$B_n = \frac{2}{L}\int_0^L 0 \cdot sin\left(\frac{n\pi x}{L}\right) dx$$

$$B_n = \frac{2}{L}\int_0^L 0$$

$$B_n = 0$$

Then plug B_n into the solution (1), we will get the solution of the heat equation as

$$u(x,t) = \sum_{n=1}^{\infty}(0)\sin(\frac{n\pi x}{L})e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$u(x,t) = 0$$

b) **Fuzzy Partial Differential Equation (FPDE)**

Solving fuzzy heat equation using Seikkala differentiability of a fuzzy-valued function:

Let $I_1 = [0,1]$ and $I_2 = [0,1]$. Consider a fuzzy heat equation

$$\frac{\partial \widetilde{U}}{\partial t} = \tilde{P} \otimes \frac{\partial^2 \widetilde{U}}{\partial x^2}$$

where $\tilde{P}$ is a fuzzy diffusivity, $\widetilde{U}(x,t)$ is fuzzy temperature at $(x,t) \in I_1 \times I_2$ and $\otimes$ is fuzzy multiplication operator. We have specific fuzzy boundary conditions $\widetilde{U}(0,t) = \widetilde{U}(1,t) = \tilde{0}$ and fuzzy initial condition $\widetilde{U}(x,0) = \tilde{C} \odot \cos\left(\pi x - \frac{\pi}{2}\right)$, where $\tilde{C}$ is a fuzzy number (An operator $\odot$ defines multiplication of a fuzzy number with a real number), and $\tilde{0}(r) = 1$ at $r = 0$ and $\tilde{0}(r) = 0$ for $r \neq 0$. As the fuzzy initial condition involves cosine function, $\frac{\partial^2 U_1}{\partial x^2} < 0, \frac{\partial^2 U_2}{\partial x^2} < 0$, the system of parametric form of heat equation

$$\frac{\partial u_1}{\partial t} = min\left\{p_1(\alpha)\frac{\partial^2 u_1}{\partial x^2}, p_1(\alpha)\frac{\partial^2 u_2}{\partial x^2}, p_2(\alpha)\frac{\partial^2 u_1}{\partial x^2}, p_2(\alpha)\frac{\partial^2 u_2}{\partial x^2}\right\},$$

$$\frac{\partial u_2}{\partial t} = min\left\{p_1(\alpha)\frac{\partial^2 u_1}{\partial x^2}, p_1(\alpha)\frac{\partial^2 u_2}{\partial x^2}, p_2(\alpha)\frac{\partial^2 u_1}{\partial x^2}, p_2(\alpha)\frac{\partial^2 u_2}{\partial x^2}\right\},$$

can be simplified as

$$\frac{\partial u_1}{\partial t} = p_2(\alpha)\frac{\partial^2 u_1}{\partial x^2}$$

$$\frac{\partial u_2}{\partial t} = p_1(\alpha)\frac{\partial^2 u_2}{\partial x^2}$$

for all $(x,t) \in I_1 \times I_2$ and all $\alpha \in [0,1]$. Subject to

$$u_i(0,t,\alpha) = u_i(1,t,\alpha) = 0$$

$$u_i(x,0,\alpha) = c_i(\alpha)cos(\pi x - \pi/2)$$

For $i = 1{,}2$. The solution is

$$u_1(x,t,\alpha) = c_1(\alpha)e^{-p_2(\alpha)\pi^2 t}\cos\left(\pi x - \frac{\pi}{2}\right) \qquad (2)$$

$$u_2(x,t,\alpha) = c_2(\alpha)e^{-p_1(\alpha)\pi^2 t}\cos(\pi x - \frac{\pi}{2}) \qquad (3)$$

for $(x,t) \in I_1 \times I_2$ and all $\alpha \in [0,1]$.

From the membership function of a triangular fuzzy number, we know that the α-level set of $\breve{a}$ is then

$$\tilde{a}_\alpha = [(1-\alpha)\alpha^L + \alpha a, (1-\alpha)a^U + \alpha a]$$

Now take fuzzy diffusivity constant as a fuzzy number $\tilde{P} = (-1{,}0{,}1)$ a triangular fuzzy number with $p_1(\alpha) = -1 + \alpha\ and\ p_2(\alpha) = 1 - \alpha$, $\alpha \in [0\ ,1]$. Let $\tilde{C} = (-1{,}0{,}1)$ as a coefficient in the fuzzy initial condition $\widetilde{U}(x,0) = \tilde{C} \odot \cos\left(\pi x - \frac{\pi}{2}\right)$. So that $\widetilde{U}(x,0) = \tilde{2} \odot \cos\left(\pi x - \frac{\pi}{2}\right)$ where $c_1(\alpha) = -1 + \alpha$ and $c_2(\alpha) = 1 - \alpha$. By substituting $p_i(\alpha)$, $c_i(\alpha), i = 1, 2$ in (2) and (3), we get the solution as

$$u_1(x,t,\alpha) = (-1+\alpha)e^{-(1-\alpha)\pi^2 t}\cos(\pi x - \frac{\pi}{2}) \qquad (4)$$

$$u_2(x,t,\alpha) = (1-\alpha)e^{-(-1+\alpha)\pi^2 t}\cos(\pi x - \frac{\pi}{2}) \qquad (5)$$

4. RESULT AND ANALYSIS

The solution obtained using the differential equation and fuzzy differential equation explained in previous section are visualized below.

a) Heat Equation

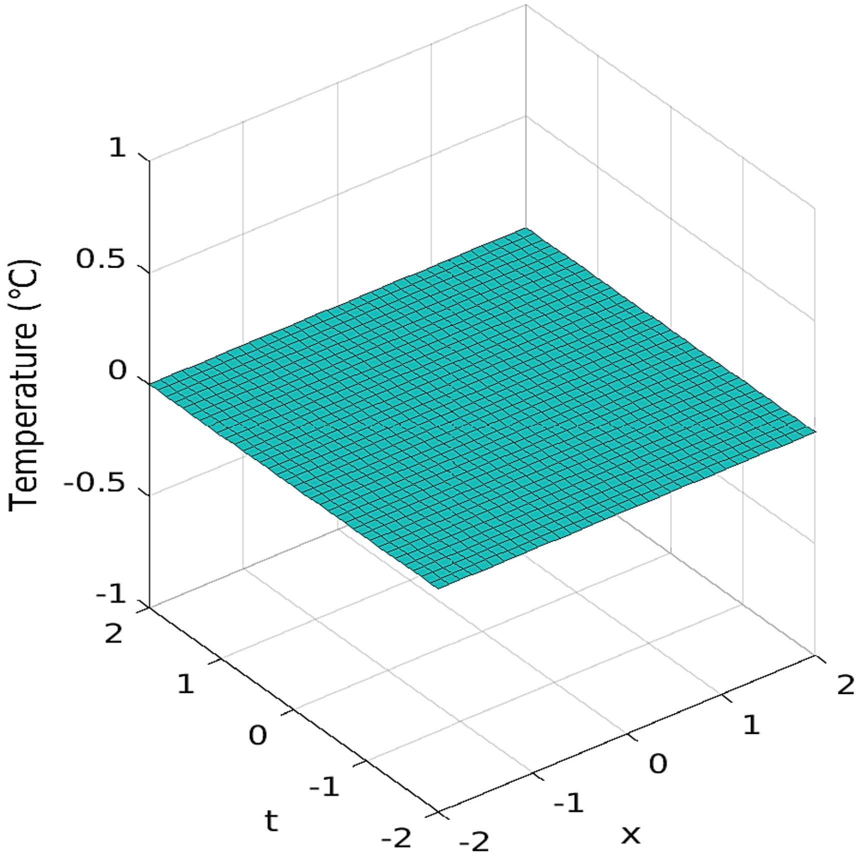

Figure 1: The solution for heat equation

Since the initial condition $f(x) = 0$, then we get the solution $u(x,t) = 0$ for all time and position of x which called the trivial solution. There are no heat sources or sinks in the rod in which the temperature in the rod is 0℃ everywhere along its length. From figure 1, it shows the temperature will be constant zero at every point in the cross section at that x since we get zero solution.

b) Fuzzy Heat Equation

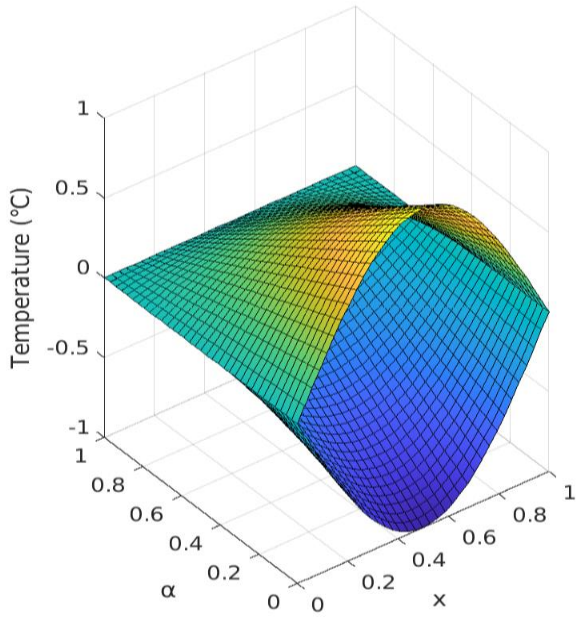

Figure 2: The solution of $u(x, t)$ when $t = 0$.

From Figure 2, the maximum diameter of the graph is 2 since the minimum and maximum temperature are -1℃ and 1℃ respectively. As α increases, the diameter of the graph decreases and it contracts to a crisp solution. Hence, the graph converges.

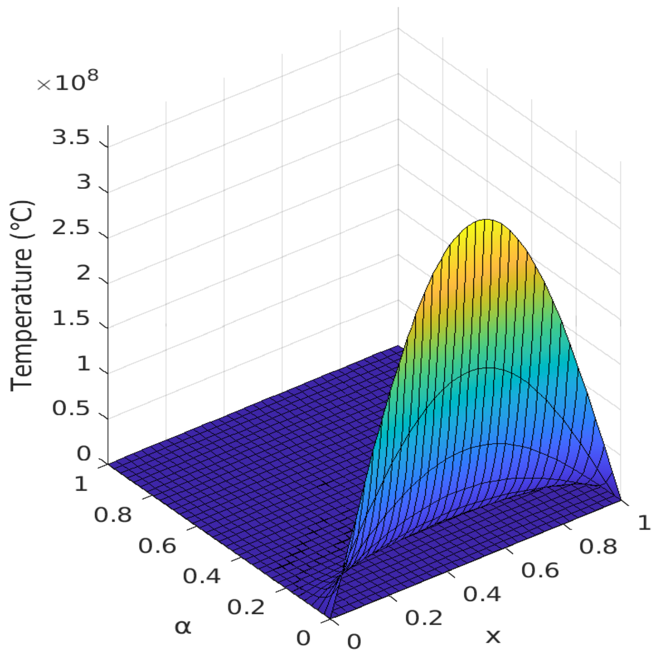

Figure 3: The solution of $u(x, t)$ when $t = 2$.

From Figure 3, the maximum diameter of the graph is 3×10^8. As α increases, the diameter of the graph decreases and it contracts to a crisp solution. Hence, the graph converges.

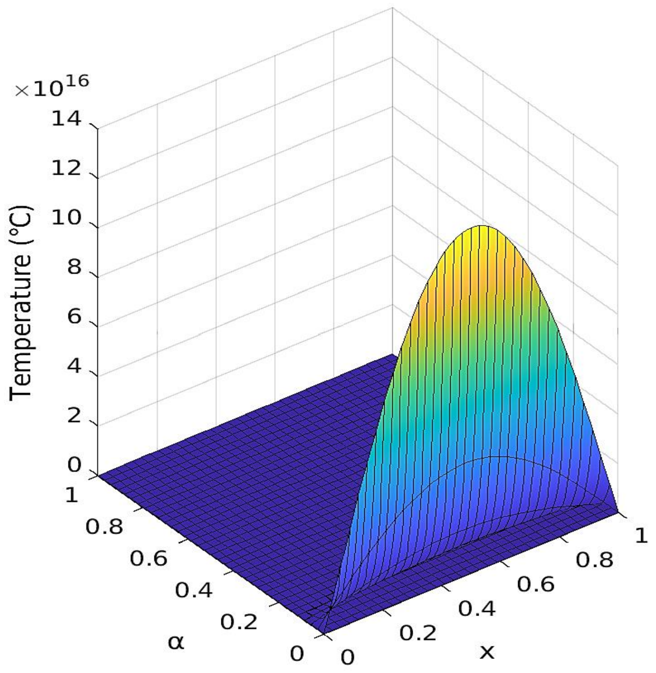

Figure 4: The solution of $u(x,t)$ when $t = 4$.

From Figure 4, the maximum diameter of the graph is 13.97×10^{16}. As α increases, the diameter of the graph decreases and it contracts to a crisp solution. Hence, the graph converges.

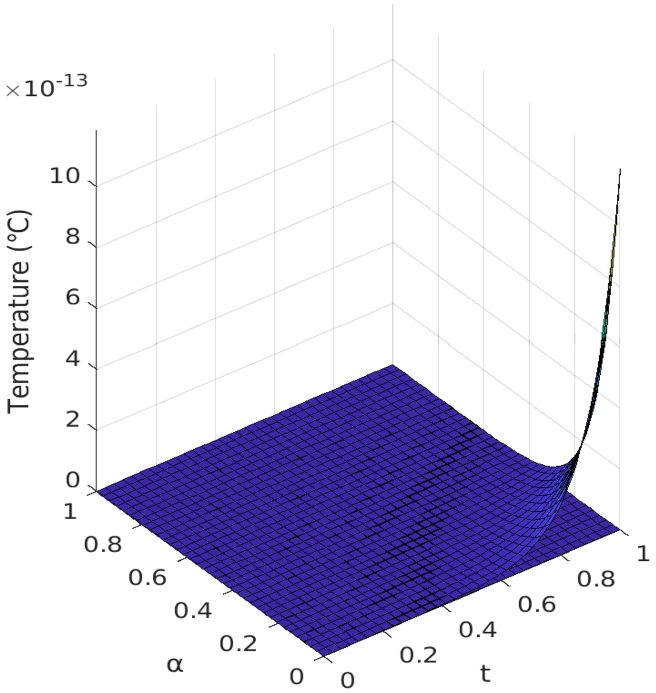

Figure 5: The solution of $u(x,t)$ when $x = 0$.

From Figure 5, the maximum diameter of the graph is 12×10^{-13}. As α increases, the diameter of the graph decreases and it contracts to a crisp solution. Hence, the graph converges.

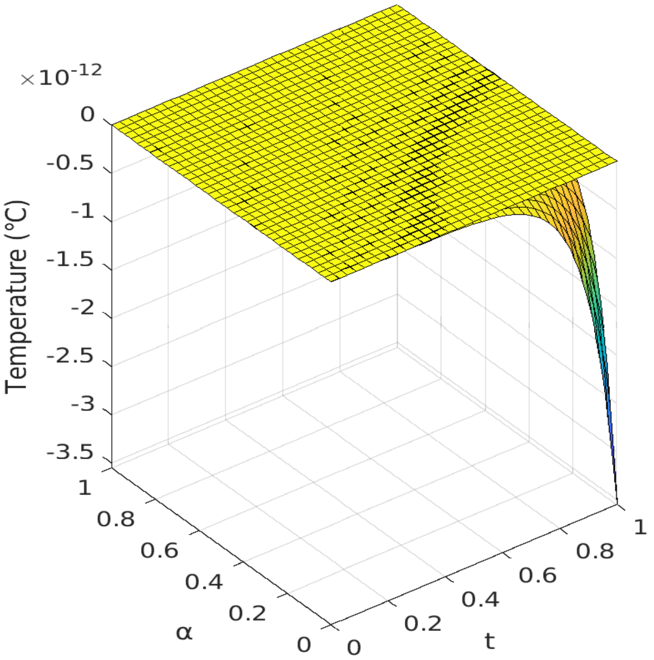

Figure 6: The solution of $u(x, t)$ when $x = 2$.

From Figure 6, the maximum diameter of the graph is 3.55×10^{-12}. As α increases, the diameter of the graph decreases and it contracts to a crisp solution. Hence, the graph converges.

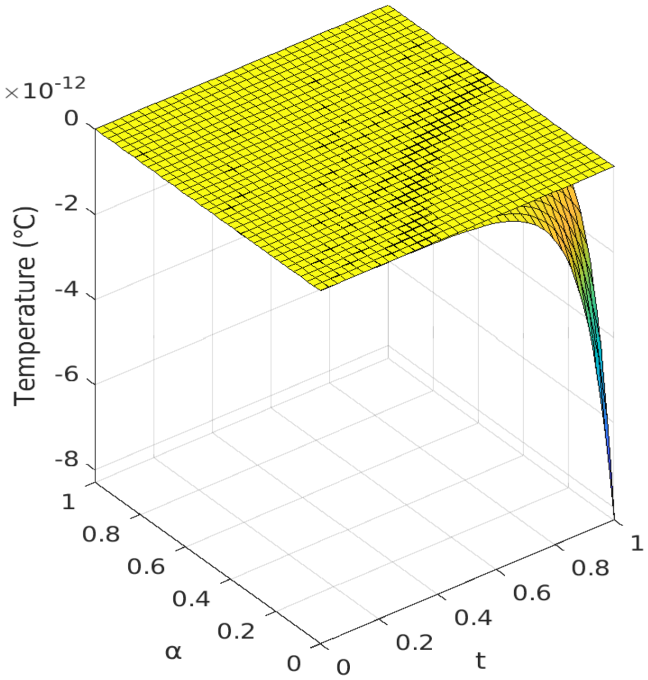

Figure 7: The solution of $u(x, t)$ when $x = 4$.

From Figure 7, the maximum diameter of the graph is 8.28×10^{-12}. As α increases, the diameter of the graph decreases and it contracts to a crisp solution. Hence, the graph converges.

5. DISCUSSION

Partial differential equation (PDE) in real world is not always perfect because it lacks information. Uncertainties occur in almost every aspect in our life, especially when dealing with real life problem. The initial value of PDEs may contain some uncertainty and fuzziness. By using fuzzy set theory, we can overcome this problem since the classical differential equation cannot cope with the uncertainty. These are the reasons of having fuzzy initial and boundary conditions in solving the heat equation. The model of uncertainty used in this fuzzy heat equation model was by the triangular fuzzy numbers. The substitution of triangular fuzzy number $(-1,0,1)$ gives the uncertainty for lower bound $-1+\alpha$ and upper bound $1-\alpha$.

There are several types of uncertainty appear in modelling and differential equation which are lack of information, abundance of information, conflicting evidence, ambiguity, measurement and belief. Lack of information is probably the most frequent cause for uncertainty. Abundance of information is due to the limited ability of human beings to perceive and process simultaneously large amounts of data. Uncertainty might also be due to conflicting evidence, there might be considerable information available pointing to a certain behavior of a system and additionally there might also be information available pointing to another behavior of the system. By ambiguity we mean a situation in which certain linguistic information, for instance, has entirely different meanings. For measurement, an "imagined" exact property cannot be measured perfectly, we have some uncertainty about the real measure and we only know the indicated measure. We know the cause of uncertainty situations in which all information available to the observer is subjective as a kind of belief in a certain situation.

By solving the heat equation, we will get zero solution for any value of x and t since the initial condition is zero. For fuzzy heat equation, we obtained the solution as in (4) and (5). The solution is proposed using Seikkala differentiability of a fuzzy-valued function. The solution is unique since Seikkala derivative provides only one solution. This means that if $f(x,t)$ and $g(x,t)$ are two different functions that satisfy the same initial boundary value problem for the heat equation, then f and g have the same form.

6. CONCLUSION

We have demonstrated the fuzzy partial differential equation for finding the solution of heat equation with uncertainty and fuzziness. The effect of fuzziness have changed the solution we obtained in heat equation from zero solution to a unique solution. Fuzzy heat equation shows a different result because of the fuzzy boundary and initial conditions. Fuzzy set theory is significant due to fuzziness and uncertainty in real-life practical application, hence, our objective to understand the application of fuzzy set theory in real world is achieved.

The relationship between partial differential equation and fuzzy partial differential equation can be seen clearly. We also provide some explanation on the type of uncertainty such as lack of information, abundance of information, conflicting evidence, ambiguity, measurement and belief. The convergence of the heat solution has been shown for different values of x and t. It can be seen that the method of Seikkala derivative is one of the methods that shows the effectiveness and accuracy in solving the heat equation. Using Seikkala method, we solve the heat equation for which it is difficult to find the solution by classical method.

REFERENCES

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APPENDIX

Coding for MATLAB Software

Coding that we are using in MATLAB software to plot 3-dimensional graph as illustrated in Section 4 are as following:

1) M-file:

```
function a = plot_x(x)
    % To find the lower and upper bound of U for value x

    syms t y; % y is alpha
    u1 = (-1 + y)*exp(-(1-y)*(pi^2)*t)*cos(pi*x - pi/2);
    u2 = (1 - y)*exp(-(-1+y)*(pi^2)*t)*cos(pi*x - pi/2);
    fsurf(u1, [0 1])
    hold on
    fsurf(u2, [0 1])
    xlabel('t'); ylabel('α'); zlabel('Temperature (℃)');
end
```

2) M-file:

```
function a = plot_t(t)
    % To find the lower and upper bound of U for value t

    syms x y; % y is alpha
    u1 = (-1 + y)*exp(-(1-y)*(pi^2)*t)*cos(pi*x - pi/2);
    u2 = (1 - y)*exp(-(-1+y)*(pi^2)*t)*cos(pi*x - pi/2);
    fsurf(u1, [0 1])
    hold on
    fsurf(u2, [0 1])
    xlabel('x'); ylabel('α'); zlabel('Temperature (℃)');
end
```

3) Command window:

```
plot_t(0)
figure(2)
plot_t(2)
figure(3)
plot_t(4)
figure(4)
plot_x(0)
figure(5)
plot_x(2)
figure(6)
plot_x(4)
```