MAT201 Advance Calculus-14.1

N.A. Abdul Rahman
Universiti Sains Malaysia

Functions of Several Variables

In this section we study functions of two or more variables from four points of view:

- verbally (by a description in words)
- numerically (by a table of values)
- algebraically (by an explicit formula)
- visually (by a graph or level curves)

Functions of Two Variables

Several examples:

Temperature: Affected by?

What else?

Functions of Two Variables

Definition A **function** f **of two variables** is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and its **range** is the set of values that f takes on, that is, $\{f(x, y) \mid (x, y) \in D\}$.

Example 2

cont'd

Wind speed (km/h)

Actual temperature (°C)

T \ v	5	10	15	20	25	30	40	50	60	70	80
5	4	3	2	1	1	0	−1	−1	−2	−2	−3
0	−2	−3	−4	−5	−6	−6	−7	−8	−9	−9	−10
−5	−7	−9	−11	−12	−12	−13	−14	−15	−16	−16	−17
−10	−13	−15	−17	−18	−19	−20	−21	−22	−23	−23	−24
−15	−19	−21	−23	−24	−25	−26	−27	−29	−30	−30	−31
−20	−24	−27	−29	−30	−32	−33	−34	−35	−36	−37	−38
−25	−30	−33	−35	−37	−38	−39	−41	−42	−43	−44	−45
−30	−36	−39	−41	−43	−44	−46	−48	−49	−50	−51	−52
−35	−41	−45	−48	−49	−51	−52	−54	−56	−57	−58	−60
−40	−47	−51	−54	−56	−57	−59	−61	−63	−64	−65	−67

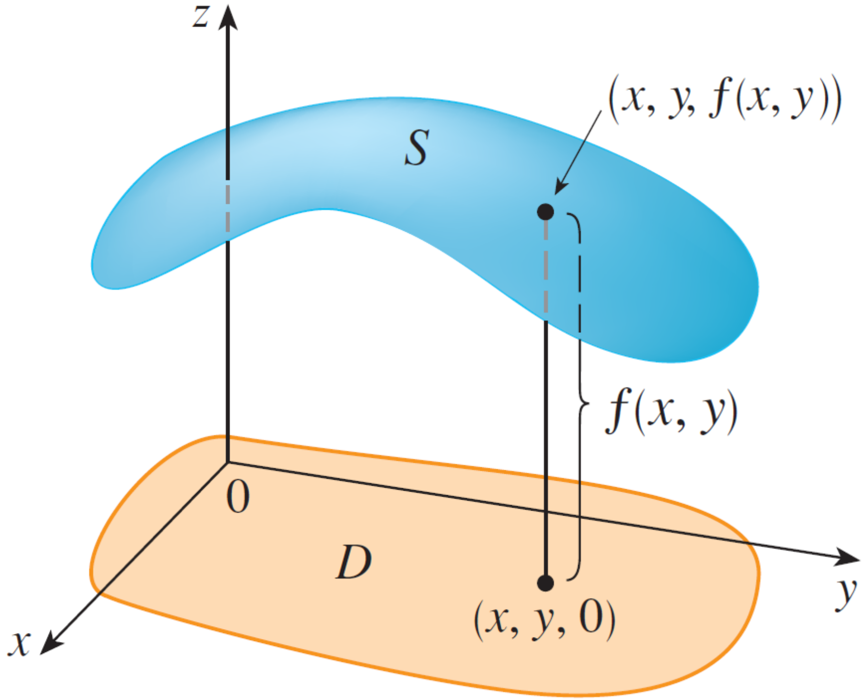
Wind-chill index as a function of air temperature and wind speed

Table 1

Graphs

Definition If f is a function of two variables with domain D, then the **graph** of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

Graphs



Graphs

The function $f(x, y) = ax + by + c$ is called as a **linear function**.

The graph of such a function has the equation

$$z = ax + by + c \qquad \text{or} \qquad ax + by - z + c = 0$$

so it is a plane.

Example 6

Sketch the graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

Solution:

$$z = \sqrt{9 - x^2 - y^2}.$$

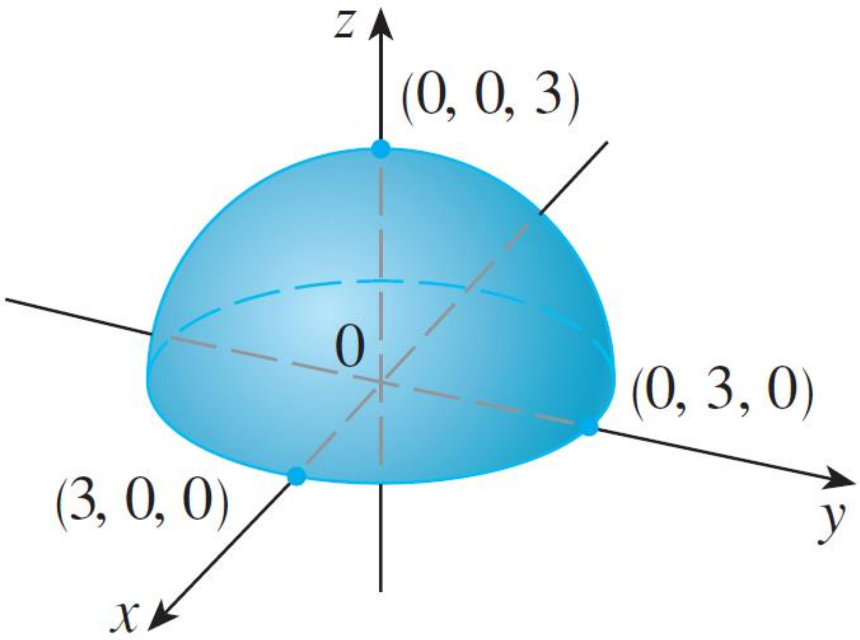
Graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$

Figure 7

Example 6

Sketch the graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

Solution:



Graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$

Level Curves

So far,

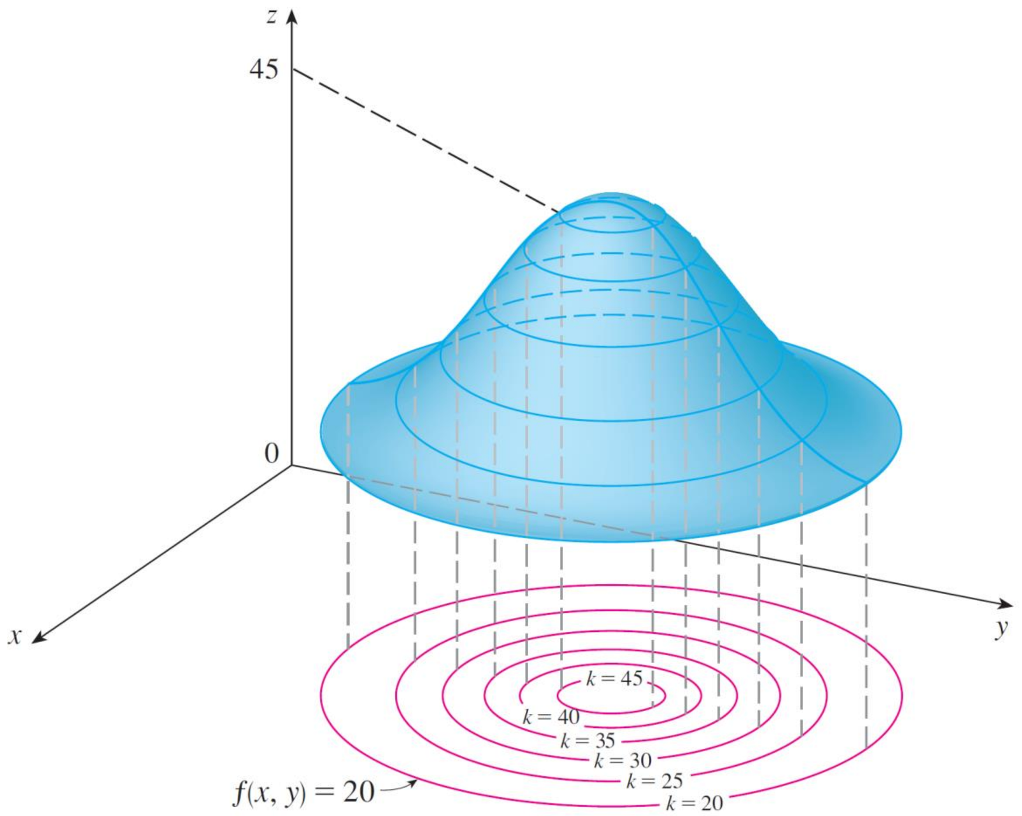
1. arrow diagrams and
2. graphs.

Next, we have level curves:

Definition The **level curves** of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).

Level Curves

You can see from figure below the relation between level curves and horizontal traces.



Level Curves

The level curves $f(x, y) = k$ are just the **traces of the graph of *f* in the horizontal plane *z* = *k* projected down to the *xy*-plane.**

The surface is **steep where the level curves are close together**. It is somewhat flatter where they are farther apart.

Level Curves

One common example of level curves occurs in topographic maps of mountainous regions, such as the map in Figure 12.

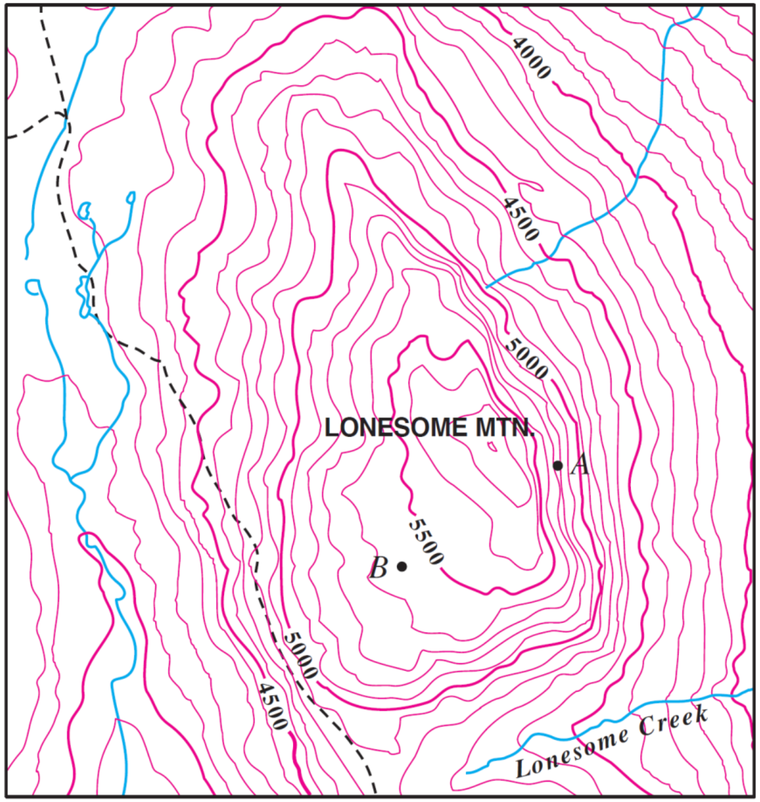
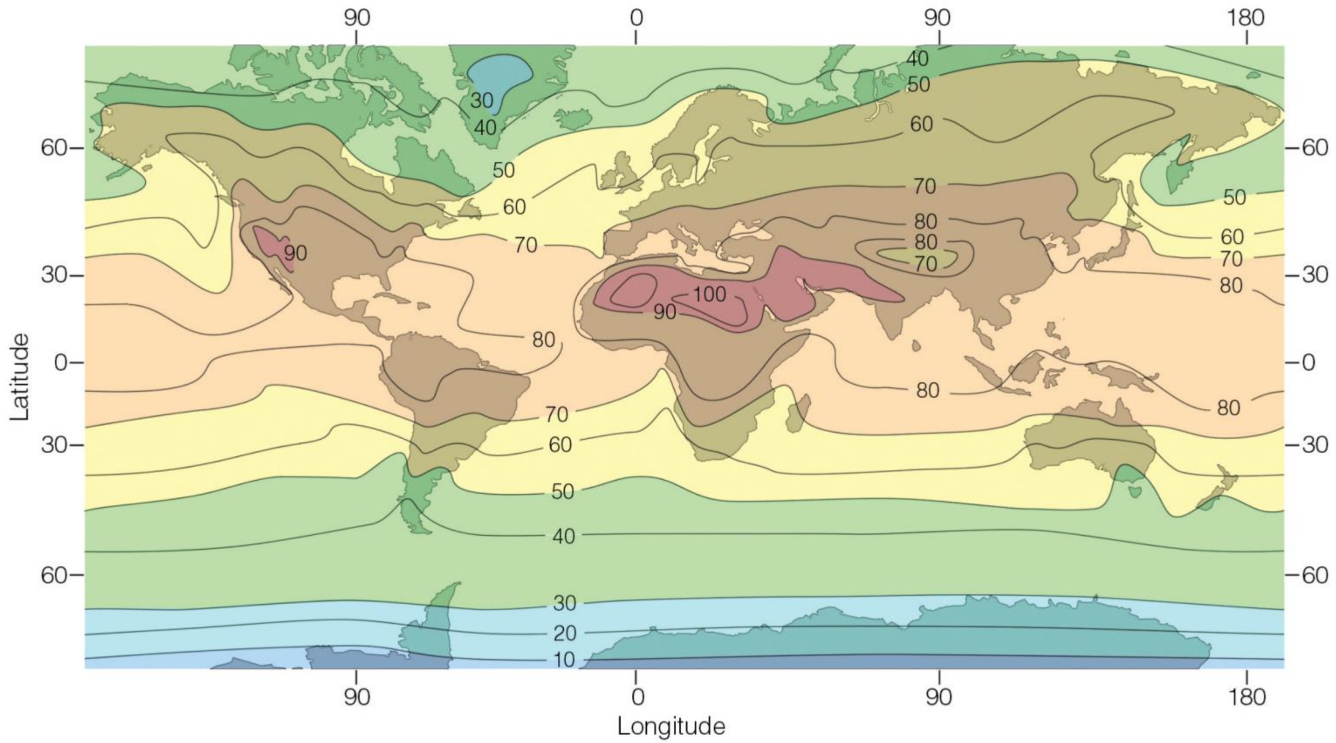


Figure 12

Level Curves

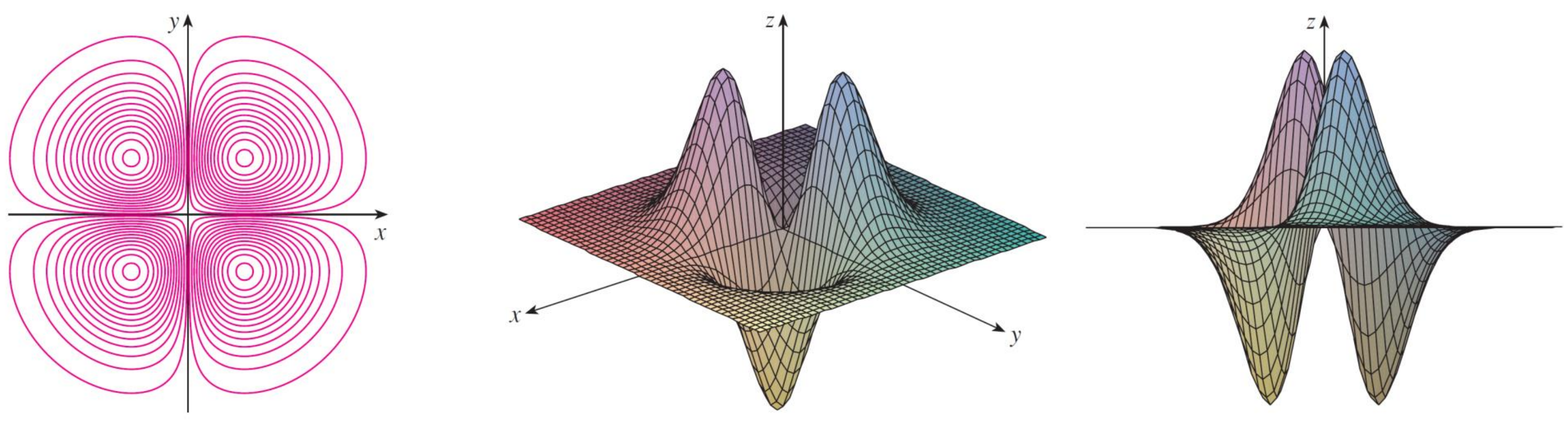
A weather map of the world indicating the average July temperatures. The isothermals are the curves that separate the colored bands.



Average air temperature near sea level in July (°F)

Level Curves

For some purposes, a contour map is more useful than a graph. It is true in estimating function values. Figure 20 shows some computer-generated level curves together with the corresponding computer-generated graphs.



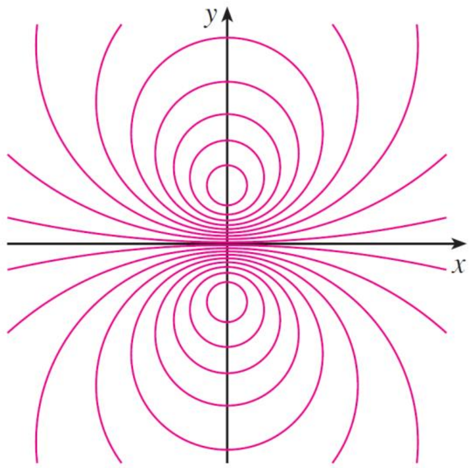
(a) Level curves of $f(x, y) = -xye^{-x^2-y^2}$

(b) Two views of $f(x, y) = -xye^{-x^2-y^2}$

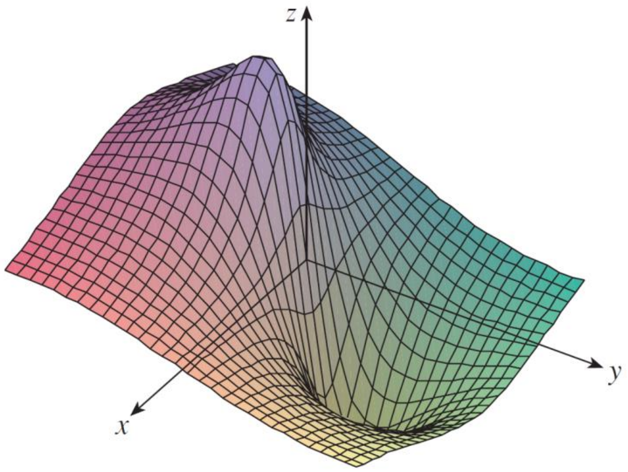
Figure 20

Level Curves

cont'd



(c) Level curves of $f(x, y) = \dfrac{-3y}{x^2 + y^2 + 1}$



(d) $f(x, y) = \dfrac{-3y}{x^2 + y^2 + 1}$

Figure 20

Notice that the level curves in part (c) crowd together near the origin. That corresponds to the fact that the graph in part (d) is very steep near the origin.

Functions of Three or More Variables

A **function of three variables**, *f*, is a rule that assigns to each ordered triple (*x*, *y*, *z*) in a domain $D \subset \mathbb{R}^3$ a unique real number denoted by *f*(*x*, *y*, *z*).

For example?

Example 14

Find the domain of *f* if

$$f(x, y, z) = \ln(z - y) + xy \sin z$$

Solution:

$$\mathbb{R}^3$$

Functions of Three or More Variables

Level surface equivalent to what in functions of three variables?

Functions of any number of variables can be considered.

A **function of *n* variables** is a rule that assigns a number $z = f(x_1, x_2, \ldots, x_n)$ to an n-tuple $(x_1, x_2, \ldots, x_n)$ of real numbers. We denote by $\mathbb{R}^n$ the set of all such n-tuples.

Functions of Three or More Variables

A company uses n different ingredients in making a food product, c_i is the cost per unit of the ith ingredient, and x_i units of the ith ingredient are used, then the total cost C of the ingredients is a function of the n variables $x_1, x_2, \ldots, x_n$:

3

$\mathbb{R}^n$

Functions of Three or More Variables

Sometimes we will use vector notation to write such functions more compactly: If $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$, we often write $f(\mathbf{x})$ in place of $f(x_1, x_2, \ldots, x_n)$.

With this notation we can rewrite the function defined in Equation 3 as

$$f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$$

where $\mathbf{c} = \langle c_1, c_2, \ldots, c_n \rangle$ and $\mathbf{c} \cdot \mathbf{x}$ denotes the dot product of the vectors $\mathbf{c}$ and $\mathbf{x}$ in V_n.

Functions of Three or More Variables

In view of the one-to-one correspondence between points $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ and their position vectors

$\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$ in V_n, we have three ways of looking at a function f defined on a subset of $\mathbb{R}^n$:

1. As a function of n real variables $x_1, x_2, \ldots, x_n$

2. As a function of a single point variable $(x_1, x_2, \ldots, x_n)$

3. As a function of a single vector variable
$\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$