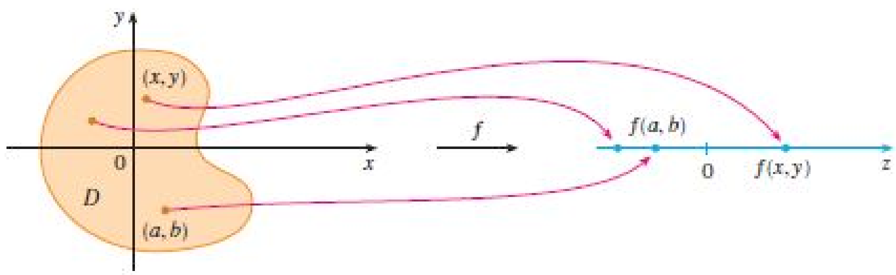
Functions of Several Variables

Definition . *A function f of two variables is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the domain of f and its range is the set of values that f takes on, that is, $\{f(x, y)|(x, y) \in D\}$.*

It is common to represent the function of of two variables as $z = f(x, y)$. The variables x and y are independent variables and $z = f(x, y)$ is the dependent variable.

A function of two variables is a function whose domain is a subset of $\mathbb{R}^2$ and whose range is a subset of $\mathbb{R}$. One way of visualizing such a function is by means of an arrow diagram (see Figure 1), where its domain (commonly denoted by D) is represented as a subset of the xy-plane.



At instance, the process of finding the domain is pretty similar for function of single variable.

Example . *For each of the following functions, evaluate $f(3, 2)$ and find the domain.*

(1) $f(x, y) = \dfrac{\sqrt{x + y + 1}}{x - 1}$

(2) $f(x, y) = x \ln(y^2 - x)$

Solution.

(1)
- $f(3, 2) = \dfrac{\sqrt{3 + 2 + 1}}{3 - 1} = \dfrac{\sqrt{6}}{2}$
- $D = \{(x, y)|x + y + 1 \geq 0,\ x \neq 1\}$

  To obtain this see that the argument inside the square root, $x + y + 1$ cannot be negative. Besides, the denominator cannot be equal to zero i.e. $x - 1 \neq 0$ or $x \neq 1$.

(2)
- $f(3, 2) = 3 \ln(2^2 - 3) = 3 \ln 1 = 0$
- $D = \{(x, y)|x < y^2\}$

  Similarly, the argument for natural log, ln must be greater or equal to zero i.e. $y^2 - x > 0$ or $x < y^2$.

Next, we try to see the situation where the domain obtained is the equation of a circle.

Example . *Find the domain and range of $g(x, y) = \sqrt{9 - x^2 - y^2}$.*

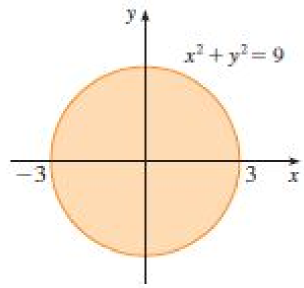
Solution. $D = \{(x, y)|9 - x^2 - y^2 \geq 0\} = \{(x, y)|x^2 + y^2 \leq 9\}$.

Recall that $x^2 + y^2 = 9$ is a circle with radius 9. Hence, the domain $D = \{(x, y)|x^2 + y^2 \leq 9\}$ tells us that the domain are enclosed within the mentioned circle.

The range of g is $\{z|z = \sqrt{9 - x^2 - y^2},\ (x,y) \in D\}$. Notice that $z \geq 0$, and $9 - x^2 - y^2 \leq 9$ implies $\sqrt{9 - x^2 - y^2} \leq 3$. So the range is

$$\{z|0 \leq z \leq 3\} = [0,3].$$

The following figure illustrate the domain obtained.



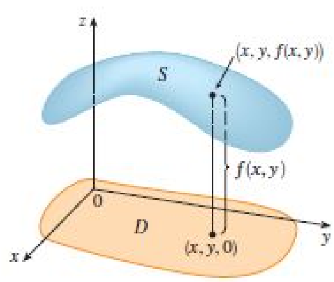
Visualizing functions

Visualizing functions gives us a lot of information regarding the problem or equation. For single variable functions, we plotted 2-dimensional graphs. For function of two variables, the graphs plotted are 3-dimensional instead, involving 3 axes, x, y and z. Another way of plotting the functions of two variables are by using level curves that are widely used in constructing geographical maps.

1. Graph

Definition . *If f is a function of two variables with domain D, then the* ***graph*** *of f is the set of all points (x,y,z) in $\mathbb{R}^3$ such that $z = f(x,y)$, and (x,y) is in D.*

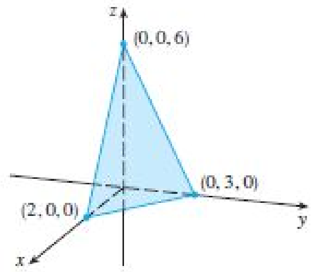
Figure below illustrate the graph for (x,y,z) with the set of domain given by D.



We can clearly see that the function $f(x,y)$, in a similar manner as function of single variable, maps (x,y) to $f(x,y)$. The graph is plotted by using the values of $(x,y,f(x,y))$.

Example . *Sketch the graph of the function $f(x,y) = 6 - 3x - 2y$ in the first octant.*

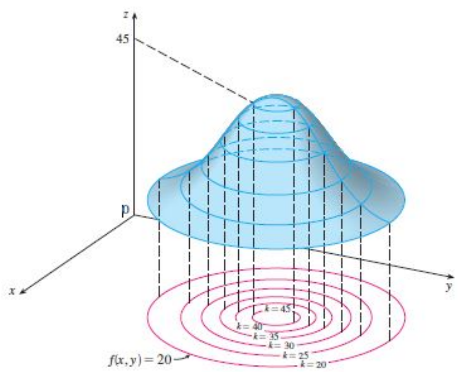
Solution. *The graph of f has the equation $z = 6 - 3x - 2y$, or $3x + 2y + z = 6$, which represents a plane. To graph the plane we first find the intercepts. Putting $y = z = 0$ in the equation, we get $x = 2$ as the x-intercept. Similarly, the y-intercept is 3 and the z-intercept is 6. This helps us sketch the portion of the graph that lies in the first octant.*

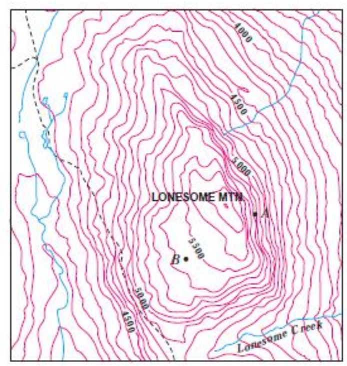


2. Level curves

Definition . *The* ***level curves*** *of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).*

Note. Figure on the left below shows an example of the level curve for several values of $f(x, y)$. Figure on the right below shows the use of level curves to represent countour or uphill or elevated areas.

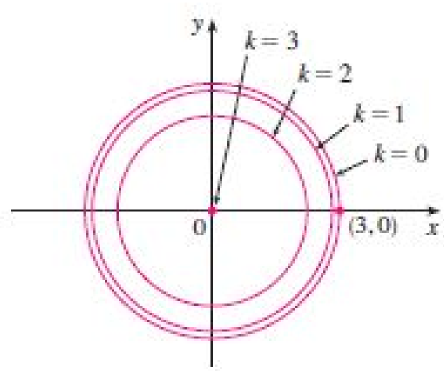




Example below will help you to visualize what is meant by level curves.

Example . *Sketch the level curves of the function $g(x, y) = \sqrt{9 - x^2 - y^2} = k$ for $k = 0, 1, 2, 3$*
Solution. *The level cureves are $g(x, y) = \sqrt{9 - x^2 - y^2} = k$ or $x^2 + y^2 = 9 - k^2$.*

This is a family of concentric circles with center $(0, 0)$ and radius $\sqrt{9 - k^2}$. The cases $k = 0, 1, 2, 3$ are shown in figure below. Try to visualize these level curves lifted up to form a surface.



Note. The values of k is constants given to replace $f(x, y)$ where k is in the range of $f(x, y)$.

Function of three or more variables

A function of three variables, f, is a rule that assigns to each ordered triple (x, y, z) in a domain $D \subset \mathbb{R}^3$ a unique real number denoted by $f(x, y, z)$. For instance, the temperature T at a point on the surface of the earth depends on the longitude x and latitude y of the point and on the time t, so we could write $T = f(x, y, t)$.

Example . *Find the domain of f if $f(x, y, z) = ln(z - y) + xy \sin z$*

Solution. *The expression for $f(x, y, z)$ is defined as long as $z - y > 0$, so the domain of f is $D = \{(x, y, z) \in \mathbb{R}^3 | z > y\}$. This is a half space consisting of all points that lie above the place $z = y$.*

1. Level surfaces.

When considering function with three varaibles, it is very hard to illustrate it in graphs. To overcome this, mathematician introduce **level surfaces** to give an insight to the behaviour of the function. The concept is pretty similar to level curves.

Definition . *The **level surfaces** of a function f of three variables are the surfaces with equations $f(x, y, z) = k$, where k is a constant (in the range of f).*

Example 2. *Find the level surfaces of the function $f(x, y, z) = x^2 + y^2 + z^2$.*

Solution. *The level surfaces are represented as $x^2 + y^2 + z^2 = k$ where $k \geq 0$. These forms a spheres with radius $\sqrt{k}$.*