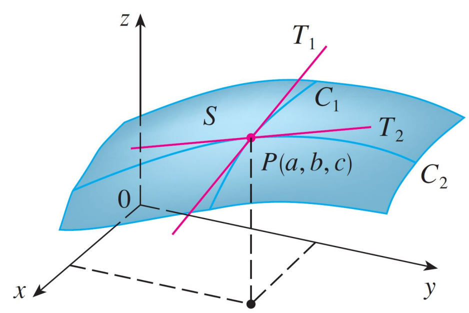
Limits and Continuity for Functions of Several variables

Definition 1. Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the **limit** of $f(x, y)$ as (x, y) approaches (a, b) is L and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for every number $\epsilon > 0$, there is a corresponding number $\delta > 0$ such that if $(x, y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$, then $|f(x, y) - L| < \epsilon$.

Definition above says that the distance between $f(x, y)$ and L can be made arbitrarily small by making the distance from (x, y) to (a, b) sufficiently small (but not 0).



Remark. If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, then, $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

Example 1. Show that $\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist.

Solution. This example is pretty straightforward. First you can check when $(x, y) \to (0, 0)$ along x-axis where $y = 0$. When $y = 0$, we have $f(x, 0) = x^2/x^2 = 1$. So,

$$f(x, y) \to 1 \qquad \text{as} \qquad (x, y) \to (0, 0) \qquad \text{along the } x\text{-axis.}$$

Similarly, you can check when $(x, y) \to (0, 0)$ along y-axis where $x = 0$. When $x = 0$, we have $f(0, y) = -y^2/y^2 = -1$. So,

$$f(x, y) \to -1 \qquad \text{as} \qquad (x, y) \to (0, 0) \qquad \text{along the } y\text{-axis.}$$

Since f has two different limits along two different lines, the given limit does not exist.

Example 2. If $f(x, y) = \frac{xy}{x^2 + y^2}$, does $\lim_{(x,y)\to(0,0)} f(x, y)$ exist?

Solution. If $y = 0$, then $f(x, 0) = \frac{0}{x^2} = 0$. Therefore,

$$f(x, y) \to 0 \quad \text{as} \quad (x, y) \to (0, 0) \text{ along the } x\text{-axis.}$$

If $x = 0$, then $f(0, y) = \frac{0}{y^2} = 0$. Therefore,

$$f(x, y) \to 0 \quad \text{as} \quad (x, y) \to (0, 0) \text{ along the } y\text{-axis.}$$

We have shown that the limit along x and y-axis approaches 0 as (x, y) approaches $(0, 0)$. We know that there are infinitely many ways for (x, y) approaches $(0, 0)$. To check on this, we shall let $(x, y) \to (0, 0)$ along any vertical lines. Then, $y = mx$, m is the slope, and

$$f(x, mx) = \frac{x(mx)}{x^2 + (mx)^2} = \frac{mx^2}{x^2 + m^2x^2} = \frac{m}{1 + m^2}$$

We see that the limit depends on the value of m instead of x and y. Let's try approaching $(0, 0)$ along the line $y = x$ and where $m = 1$.

$$f(x, x) = \frac{x^2}{x^2 + x^2} = 1/2.$$

Therefore,

$$f(x, y) \to 1/2 \quad \text{as} \quad (x, y) \to (0, 0) \text{ along } y = x.$$

Since we have obtained different limits along different paths, the given limit does not exist.

Similar to functions of a single variable, the calculation of limits for functions of two variables can be greatly simplified by the use of properties of limits. The Limit Laws for single variable functions can be extended to functions of two variables.

$$\lim_{(x,y)\to(a,b)} x = a, \qquad \lim_{(x,y)\to(a,b)} y = b, \qquad \lim_{(x,y)\to(a,b)} c = c.$$

Example 3. Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2 + y^2}$ if it exists.

Solution. As in previous examples, we could show that the limit along any line through the origin is 0. This doesn't prove that the given limit is 0, but the limits along the parabolas $y = x^2$ and $x = y^2$ also turn out to be 0, so we begin to suspect that the limit does exist and is equal to 0.

Let $\epsilon > 0$. We want to find $\delta > 0$ such that

$$if \qquad 0 < \sqrt{x^2 + y^2} < \delta \qquad \text{then} \qquad \left| \frac{3x^2y}{x^2 + y^2} - 0 \right| < \epsilon$$

that is

$$if \qquad 0 < \sqrt{x^2 + y^2} < \delta \qquad \text{then} \qquad \frac{3x^2|y|}{x^2 + y^2} < \epsilon$$

But $x^2 \leq x^2 + y^2$ since $y^2 \geq 0$, so $x^2/(x^2 + y^2) \leq 1$, and therefore

$$\frac{3x^2|y|}{x^2 + y^2} \leq 3|y| = 3\sqrt{y^2} \leq 3\sqrt{x^2 + y^2}.$$

Thus, if we choose $\delta = \epsilon/3$ and let $0 < 3\sqrt{x^2 + y^2} < \delta$, then

$$\left| \frac{3x^2y}{x^2 + y^2} - 0 \right| \leq 3\sqrt{x^2 + y^2} < 3\delta = 3\,(\epsilon/3) = \epsilon.$$

Hence,

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2 + y^2} = 0.$$

Continuity

Definition 2. A function f of two variables is called **continuous** at (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

We say that f is **continuous** on D if f is continuous at every point (a,b) in D.

Example 4. Let

$$f(x,y) = \begin{cases} \dfrac{3x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

We know f is continuous for $(x,y) \neq (0,0)$ since it is equal to a rational function there. In the previous example, we see that

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0 = f(0,0).$$

Therefore, f is continuous at $(0,0)$ and so it is continuous on $\mathbb{R}^2$. Example not continuous (with graph)

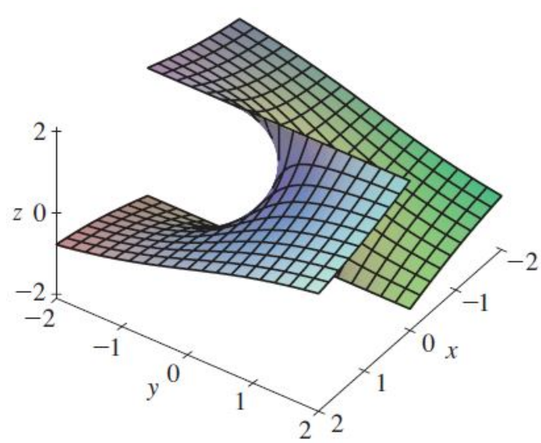
Anologous to functions of single variable, composition is another way of combining two continuous functions to get a third. In fact, it can be shown that if f is a continuous function of two variables and g is a continuous function of a single variable that is defined on the range of f, then the composite function $h = g \circ f$ defined by $h(x,y) = g(f(x,y))$ is also a continuous function.

Example 5. Where is the function $h(x,y) = \arctan(y/x)$ continuous?

Solution. The function $f(x,y) = y/x$ is a rational function and therefore continuous except on the line $x = 0$. The function $g(t) = \arctan(t)$ is continuous everywhere. So, the composite function

$$g(f(x,y)) = \arctan(y/x) = h(x,y)$$

is continuous except whre $x = 0$. The graph below shows the break in the graph h above the y-axis.



Functions of three or more variables

The notation

$$\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = L.$$

means that the values of $f(x,y,z)$ approach the number L as the point (x,y,z) approaches the point (a,b,c) along any path in the domain of f. Because the distance between two points (x,y,z) and (a,b,c) in $\mathbb{R}^3$ is given by $\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}$, we can write the precise defintion of limit for functions with two or more variables as follows using vectors.

Definition 3. If f is defined on a subset D of $\mathbb{R}^n$, then $\lim_{\tilde{x}\to\tilde{a}} f(\tilde{x}) = L$ means that for every number $\epsilon > 0$, there is a corresponding number $\delta > 0$ such that,

$$\text{if} \qquad \tilde{x} \in D \qquad \text{and} \qquad 0 < |\tilde{x}-\tilde{a}| < \delta \qquad \text{then} \qquad |f(\tilde{x}) - L| < \epsilon.$$

Notice that if $n = 1$, then $\tilde{x} = x$ and $\tilde{a} = a$, and the definition above is just the definition of a limit for functions of a single variable. For the case $n = 2$, we have $\tilde{x} =< x, y >$, $\tilde{a} =< a, b >$, and $|\tilde{x}-\tilde{a}| = \sqrt{(x-a)^2+(y-b)^2}$. If $n = 3$, then $\tilde{x} =< x, y, z >$ and $\tilde{a} =< a, b, c >$, we have the definition of a limit for functions of three variables. In each case, the definition of continuity can be written as

$$\lim_{\tilde{x}-\tilde{a}} f(\tilde{x}) = f(\tilde{a}).$$