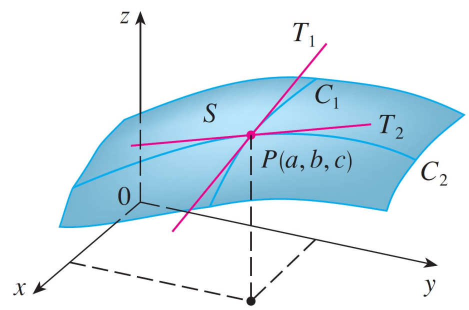
Partial Derivatives

In the case of functions with two variables, the derivative of a function is quite complicated. When we consider the function in its graphical form, a surface, we might ask about "how steep is the surface?" and this question is somehow obvious question about derivatives. To answer on the 'steepness' however, is not simple.

First, let us consider the surface S of function $f(x, y)$.



If $f(a, b) = c$, then the point $P(a, b, c)$ lies on S. By fixing $y = b$, we are restricting our attention to the curve C_1 in which the vertical plane intersects S. (In other words, C_1 is the trace of S in the plane $y = b$.) Likewise, the vertical plane $x = a$ intersects S in a curve C_2. Both of the curves C_1 and C_2 pass through the point P.

Notice that the curve C_1 is the graph of the function $g(x) = f(x, b)$, so the slope of its tangent T_1 at P is $g'(a) = f_x(a, b)$. The curve C_2 is the graph of the function $h(y) = f(a, y)$, so the slope of its tangent at is $h'(a) = f_y(a, b)$.

Thus, the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ can be interpreted geometrically as the slopes of the tangent lines at $P(a, b, c)$ to the traces C_1 and C_2 of S in the planes $y = b$ and $x = a$.

In general, we let only x vary while keeping fixed y, say $y = b$, where b is a constant. Then, we are really considering a function of a single variable x, namely, $g(x) = f(x, b)$. If g has a derivative at a, then we call it the partial derivative of f with respect to x at (a, b) and denote it by $f_x(a, b)$.

$$f_x(a, b) = g'(a) \qquad \text{where} \qquad g(x) = f(x, b)$$

and by the definition of derivative,

$$g'(a) = \lim_{h \to 0} \frac{g(a + h) - g(a)}{h}$$

Therefore,

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}.$$

Analogously, the partial derivative of f with respect to y at (a, b),

$$f_y(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}.$$

If we now let the point (a,b) vary, f_x and f_y become functions of two variables. Therefore,

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h},$$
$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}.$$

If you have read about partial derivatives, you can see there are a lot of alternative notations, some of which are:

$$(1)\ f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1(f) = D_x f,$$
$$(2)\ f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = f_2 = D_2(f) = D_y f.$$

To compute partial derivatives, you need to recall techniques from single variable calculus since you are treating the function with respect to a single variable at once. As a matter of facet, we have the following rules for computing partial derivatives.

Example 1. If $f(x,y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2,1)$ and $f_y(2,1)$.
Solution. Holding y constant and differentiating with respect to x, we have

$$f_x(x,y) = 3x^2 + 2xy^3$$
$$\text{and so} \qquad f_x(2,1) = 3.2^2 + 2.2.1^3 = 16.$$

Holding x constant and differentiating with respect to y, we have

$$f_y(x,y) = 3x^2y^2 - 4y$$
$$\text{and so} \qquad f_y(2,1) = 3.2^2.1^2 - 4.1 = 8.$$

Remark. Partial derivatives can also be interpreted as *rates of change*. If $z = f(x,y)$, then $\partial z/\partial x$ represents the rate of change of z with respect to x when y is fixed. Similarly, $\partial z/\partial y$ represents the rate of change of z with respect to y when x is fixed.

Example 2

Example 2. If $f(x,y) = \sin\left(\frac{x}{1+y}\right)$, calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$,
Solution. Using the Chain Rule for functions of one variable, we have

$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{1+y}\right) . \frac{\partial}{\partial x}\left(\frac{x}{1+y}\right) = \cos\left(\frac{x}{1+y}\right) . \frac{1}{1+y},$$
$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{1+y}\right) . \frac{\partial}{\partial x}\left(\frac{x}{1+y}\right) = \cos\left(\frac{x}{1+y}\right) . \frac{1}{1+y}$$

Functions of more than two variables

Partial derivative for a function of two variables can be easily extend to functions of three or more variables. For example,

$$f_x(x, y, z) = \lim_{h \to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}$$

and it is computed by regarding y and z as constants and differentiating $f(x, y, z)$ with respect to x. If $u = f(x, y, z)$, we denote partial derivative of f with respect to x as $f_x = \partial u/\partial x$.
In general, for functions of n variables, we have the partial derivatives with respect to the i-th variable, let say x_i, as follows.

$$\frac{\partial u}{\partial x_1} = \lim_{h \to 0} \frac{f(x1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h}$$

Example 3. Find f_x, f_y, and f_z if $f(x, y, z) = e^{xy} \ln z$.

Solution. Holding y and z constant and differentiating with respect to x, we have $f_x = ye^{xy} \ln z$. Similarly, $f_y = xe^{xy} \ln z$ and $f_z = \dfrac{e^{xy}}{z}$.

Higher Derivatives

If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables, so we can consider their partial derivatives $(f_x)_x$, $(f_x)y$, $(f_y)_x$, and $(f_y)_y$, which are called the second partial derivatives of f. If $z = f(x, y)$. These are some commonly used notations for second partial derivative.

$$\begin{aligned}
(f_x)_x &= f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}, \\
(f_x)_y &= f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}, \\
(f_y)_x &= f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}, \\
(f_y)_y &= f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}.
\end{aligned}$$

Example 4. Find the second partial derivative of $f(x, y) = x^3 + x^2y^3 - 2y^2$.

Solution. From Example 1, we know that

$$f_x(x, y) = 3x^2 + 2xy^3, \qquad f_y(x, y) = 3x^2y^2 - 4y.$$

Thus,

$$\begin{aligned}
f_{xx} &= \frac{\partial}{\partial x} 3x^2 + 2xy^3 = 6x + 2y^3, \\
f_{xy} &= \frac{\partial}{\partial y} 3x^2 + 2xy^3 = 6xy^2
\end{aligned}$$

and

$$f_{yy} = \frac{\partial}{\partial y} 3x^2y^2 - 4y = 6x^2y - 4,$$
$$f_{yx} = \frac{\partial}{\partial x} 3x^2y^2 - 4y = 6xy^2$$

Notice that $f_{xy} = f_{yx}$. This is not just a coincidence. It turns out that the mixed partial derivatives f_{xy} and f_{yx} are equal for most functions that one meets in practice. The following theorem, which was discovered by the French mathematician Alexis Clairaut (1713–1765), gives conditions under which we can assert that $f_{xy} = f_{yx}$.

Theorem 1. (Clairaut's Theorem) *Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then*

(1) $$f_{xy}(a, b) = f_{yx}(a, b).$$

This theorem is also defined for partial derivatives of order 3 and higher. For example, $f_{xyy} = f_{yxy} = f_{yyx}$.

Example 5. Calculate f_{xxyz} if $f(x, y, z) = \sin(3x + yz)$.

Solution.

$$f_x = 3\cos(3x + yz)$$
$$f_{xx} = -9\sin(3x + yz),$$
$$f_{xxy} = -9zcos(3x + yz),$$
$$f_{xxyz} = -9cos(3x + yz) + 9yzsin(3x + yz).$$