Tangent Planes and Linear Approximation

One of the most important ideas in single-variable calculus is that as we zoom in toward a point on the graph of a differentiable function, the graph becomes indistinguishable from its tangent line and we can approximate the function by a linear function. We shall utilize the same idea when considering function with more than two variables particularly on function with three variables.

As we zoom in toward a point on a surface that is the graph of a differentiable function of two variables, the surface looks more and more like a plane (its tangent plane) and we can approximate the function by a linear function of two variables.

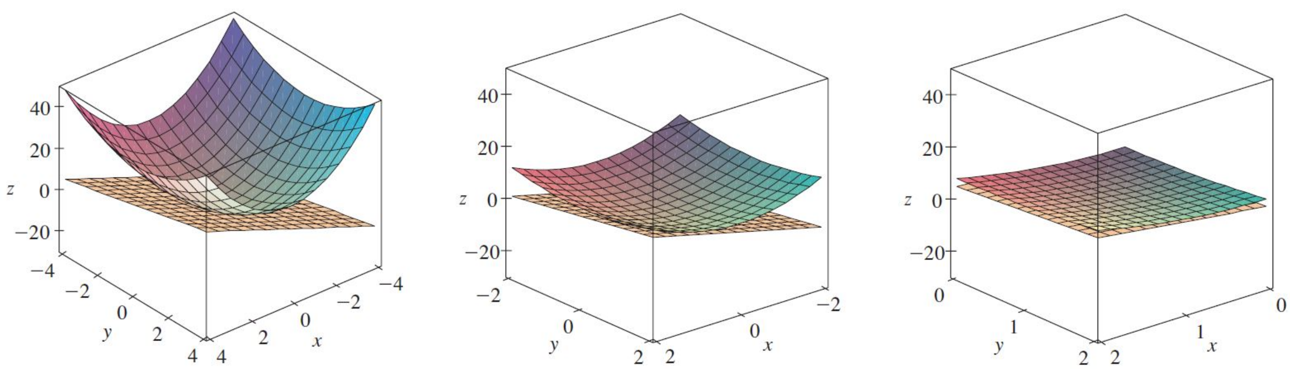
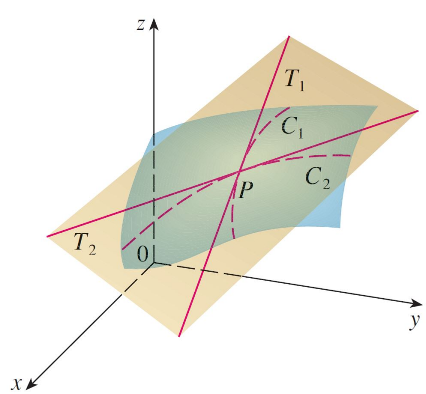


Figure 1. The graphs plotted are function $z = 2x^2 + y^2$. Notice that the tangent plane become almost parallel to the surface as we zoom in (from left to right).

Can you relate this on why the earth seems like flat for us but the photos from satellite shows that the earth is in fact a globe?

Tangent Planes

To see how tangent planes relate to partial derivatives, consider the following figure.



Suppose a surface has equation $z = f(x, y)$, where f has continuous first partial derivatives, and let $P(x_0, y_0, z_0)$ be a point on S. As in the preceding section, let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S. Then the point lies on both C_1 and C_2. Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P. Then the **tangent plane** to the surface S at the point P is defined to be the plane that contains both tangent lines T_1 and T_2.

On the other hand, you can also see that both tangent line of C_1 and C_2 lie on the tangent plane. So, you might think that the tangent plane to S at P consisting of all possible tangent lines at P to curves that lie on S and pass through P.

From Calculus 1, we learn that the equation of any plane passing through point $P(x_0, y_0, z_0)$ is given by

$$A(x - x_0) + B(y - y_0) + C(z - z_0)$$

By some manipulation, we have

$$z - z_0 = a(x - x_0) + b(y - y_0), \qquad a = -A/C \text{ and } b = -B/C.$$

To find a, we consider the intersection between the tangent plane at P with the plane $y = y_0$ (like $T1$, if you consider the previous graph). Substituting $y = y_0$ into the previous equation, we have

$$z - z_0 = a(x - x_0).$$

Recall that $\dfrac{z - z_0}{x - x_0} = f_x$, so $a = f_x(x_0, y_0)$. Similarly when we consider the interception between tangent plane at P with the plane $x = x_0$, we will have $b = f_y(x_0, y_0)$.

Therefore, we arrive at the following conclusion.

Suppose f has continuous partial derivatives. An **equation of the tangent plane** to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x - x_0) + f_y(y - y_0). \tag{1}$$

Example 1. Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.
Solution. Let $f(x, y) = 2x^2 + y^2$. Then,

$$f_x(x, y) = 4_x \qquad f_y(x, y) = 2y$$
$$f_x(1, 1) = 4 \qquad f_y(1, 1) = 2.$$

Using the equation of tangent plane, at $(1, 1, 3)$, we have

$$z - 3 = 4(x - 1) + 2(y - 1)$$
$$\text{or} \qquad z = 4x + 2y - 3.$$

Linear Approximations

From Example 1, we find that the equation of the tangent plane to the function $f(x, y) = 2x^2+y^2$ at $(1, 1, 3)$ is $z = 4x + 2y - 3$. So, the function $L(x, y) = 4x + 2y - 3$ can be used as an approximation of $f(x, y)$ at (x, y) near $(1, 1)$. **The function L is called the linearization of f at** $(1, 1)$. For example, consider the point $(1.1, 0.95)$, we have

$$f(1.1, 0.95) \approx 4(1.1) + 2(0.95) - 3 = 3.3$$

which is pretty close to $f(1.1, 0.95) = 2(1.1)^2 + (0.95)^2 = 3.3225$. Bear in mind that this only works if the point considered is close to $(1, 1)$.

From Equation 1, if we consider (x_0, y_0)=(a, b) and $z_0 = f(a, b)$ (point $(a, b, f(a, b))$, we have

$$z = f(a, b) + f_x(x - x_0) + f_y(y - y_0) = L(x, y)$$

since z, the equation of tangent to the plane is also the linear function to whose graph is tangent to the plane given by $L(x, y)$. Therefore, the approximation

$$f(x, y) \approx f(a, b) + f_x(x - x_0) + f_y(y - y_0)$$

is called the **linear approximation** or the **tangent plane approximation** of f at (a, b).

Definition 1. If $z = f(x, y)$, then f is differentiable at (a, b) if Δz can be expressed in the form

$$\Delta z - f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where ϵ_1 and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

From the definition, we can say that the tangent plane approximates the graph of f better near the point of tangency. At times, checking the differentiability using this definition is quite hard. The next theorem provides a more convenient sufficient condition for differentiability.

Theorem 1. *If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).*

Example 2. Show that $f(x, y) = xe^{xy}$ is differentiable at $(1, 0)$ and find its linearization there. Then use it to approximate $f(1.1, -0.1)$.
Solution. The partial derivatives are

$$\begin{aligned} f_x(x, y) &= e^{xy} + xye^{xy} \qquad & f_y(x, y) &= x^2 e^{xy} \\ f_x(1, 0) &= 1 & f_y(0, 1) &= 1. \end{aligned}$$

Both f_x and f_y are continuous, so they are differentiable according to the Definition 1. Thus, the linearization is

$$\begin{aligned} L(x, y) &= f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0) \\ &= x + y \end{aligned}$$

The corresponding linear approximation is

$$\begin{aligned} & f(x, y) = e^{xy} \approx x + y \\ so, \qquad & f(1.1, -0.1) \approx 1.1 - 0.1 = 1 \end{aligned}$$

which is quite close to the actual value, 0.98542.