Implicit Differentiation

The Chain Rule can be used to give a more comprehensive description of the process of implicit differentiation involving function of several variables. We suppose that an equation of the form $F(x, y)$ defines y implicitly as a differentiable function of x, that is, $y = f(x)$, where $F(x, f(x)) = 0$ for all x in the domain of f. If F is differentiable, we can apply Case 1 of the Chain Rule to differentiate both sides of the equation $F(x, y) = 0$ with respect to x. Since **both** x **and** y **are functions of** x, by using Chain Rule, we obtain

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

But $\frac{dx}{dx} = 1$, so if $\frac{\partial F}{\partial y} \neq 0$, we solve for $\frac{dy}{dx}$ and obtain

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y} \tag{1}$$

To derive Equation (1), assumed that $F(x, y) = 0$ defines y implicitly as a function of x. The **Implicit Function Theorem** gives condition which this assumption is valid. It takes that if F is defined on a disk containing (a,b), where $F(a, b) = 0$, $F_y(a, b) \neq 0$, and F_x and F_y are continuous on the disk, then the equation $F(x, y) = 0$ defines y as a function of x near the point (a, b) and the derivative of this function is given by Equation (1).

Example 1. Find y' if $x^3 + y^3 = 6xy$.

Solution. The equation can be written as

$$F(x, y) = x^3 + y^3 - 6xy = 0,$$

So, from Equation (1), we have

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2 - 6y}{3y^2 - 6x} = -\frac{x^2 - 2y}{y^2 - 2x}$$

Now we suppose that z is given implicitly as a function $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0$. This means that $F(x, y, f(x, y)) = 0$ for all (x, y) in the domain of f. If F and f are differentiable, then we can use the Chain Rule to differentiate the equation $F(x, y, z) = 0$ with respect to x as follows:

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

But

$$\frac{\partial x}{\partial x} = 1 \text{ and } \frac{\partial y}{\partial x} = 0.$$

($\frac{\partial y}{\partial x} = 0$ means you are differentiating a constant with respect to x since y, like x is also an independent variable)

So,

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0,$$

If $\dfrac{\partial F}{\partial z} \neq 0$, we solve for $\dfrac{\partial z}{\partial x}$ and obtain

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}.$$

Similarly for $\dfrac{\partial z}{\partial y}$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}. \tag{2}$$

Again, a version of the **Implicit Function Theorem** gives conditions under which our assumption is valid: If F is defined within a sphere containing (a, b, c), where $F(a, b, c) = 0$, $F_z(a, b, c) \neq 0$, and F_x, F_y and F_z are continuous inside the sphere, then the equation $F(x, y, z) = 0$ defines z as a function of x and y near the point (a, b, c) and this function is differentiable, with partial derivatives given by Equation (2).

Example 2. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz - 1$.

Solution. Let $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1$. Then from Equation 2, we have

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + 6yz}{3z^2 + 6xy} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$
$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3y^2 + 6xz}{3z^2 + 6xy} = -\frac{y^2 + 2xz}{z^2 + 2xy}.$$