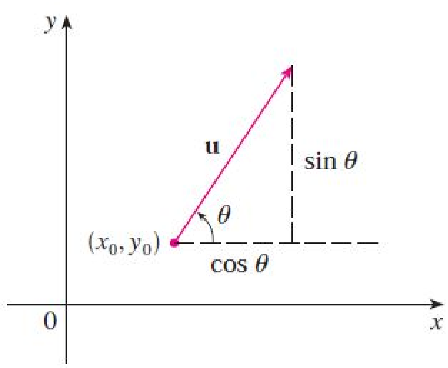
Directional Derivatives

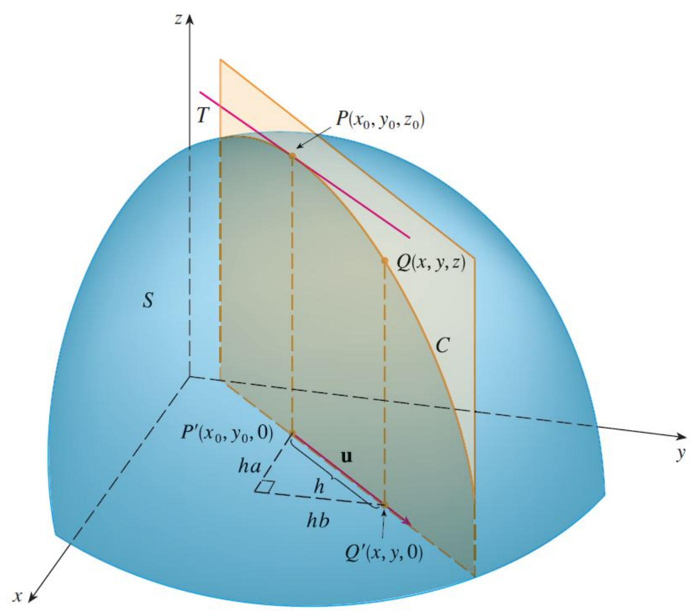
Recall that if $z = f(x, y)$, then the partial derivatives f_x and f_y are defined as

$$f_x(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$
$$f_y(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

and represents the rates of changes of z in the x and y-directions, that is, in the directions of the unit vector **i** and **j**. But, what if we want to consider the rate of change of z in the direction of an arbitrary unit vector **u**$= \langle a, b\rangle$?

To do this, we consider the surface S with equation $z = f(x, y)$ (the graph of f) and we let $z_0 = f(x_0, y_0)$. Then the point $P(x_0, y_0, z_0)$ lies on S. The vertical plane that passes through P in the direction of **u** intersects S in a curve C. (See Figure 3.) The slope of the tangent line T to C at the point P is the rate of change of z in the direction of **u**.





If $Q(x, y, z)$ is another point on C and P', Q' are the projections of P, Q on the xy-plane, then the vector $\vec{P'Q'}$ is parallel to **u** and so

$$\vec{P'Q'} = h\mathbf{u} = \langle ha, hb\rangle$$

for some scalar h. Therefore $x - x_0 = ha$, $y - y_0 = hb$, so $x = x_0 + ha$, $y = y_0 + hb$, and

$$\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}.$$

If we take the limit as $h \to 0$, we obtain the rate of change of z (with respect to distance) in the direction of **u**, which is called the directional derivative of f in the direction of **u**.

Definition 1. The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b\rangle$ is

$$D_u f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

When we compute the directional derivative of a function defined by a formula, we generally use the following theorem.

Theorem 1. *If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b\rangle$ and*

$$D_u f(x, y) = f_x(x, y)a + f_y(x, y)b$$

Proof. If we define a function g of the single variable h by

$$g(h) = f(x_0 + ha, y_0 + hb)$$

then, by the definition of a derivative, we have

$$\begin{aligned} g'(0) &= \lim_{h\to 0} \frac{g(h) - g(0)}{h} = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} \\ &= D_u f(x_0, y_0) \end{aligned} \tag{1}$$

On the other hand, we can write $g(h) = f(x, y)$, where $x = x_0 + ha$, $y = y_0 + hb$, so the Chain Rule gives

$$g'(h) = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh} = f_x(x, y)a + f_y(x, y)b$$

If we now put $h = 0$, the $x = x_0, y = y_0$, and

$$g'(0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b \tag{2}$$

Comparing Equations (1) and (2), we see that

$$D_u f(x_0, y_0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b$$

□

Sometimes, we will work with direction in terms of θ which is angles from the positive x-axis, then we may write **u**$= \langle \cos\theta, \sin\theta\rangle$. Hence, from Theorem 1,

$$D_u f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta. \tag{3}$$

Example 1. Find the directional derivative $D_u f(x, y)$ if $f(x, y) = x^3 - 3xy + 4y^2$ and **u** is the unit vector given by the angle $\theta = \pi/6$. What is $D_u f(1, 2)$?

Solution. From Equation (3),

$$\begin{aligned} D_u f(x, y) &= f_x(x, y)\cos\frac{\pi}{6} + f_y(x, y)\sin\frac{\pi}{6} \\ &= (3x^2 - 3y)\frac{\sqrt{3}}{2} + (-3x + 8y)\frac{1}{2} \\ &= \frac{1}{2}[3\sqrt{3}x^2 - 3x + (8 - 3\sqrt{3})y] \end{aligned}$$

Therefore

$$D_u f(1, 2) = \frac{1}{2}[3\sqrt{3}(1)^2 - 3(1) + (8 - 3\sqrt{3})(2)] = \frac{13 - 3\sqrt{3}}{2}$$

The Gradient Vector

From Theorem 1, we see that te equation for directional derivatives can also be written as a dot product of two vectors,

$$D_u f(x,y) = \langle f_x(x,y)a + f_y(x,y)\rangle \cdot \langle a,b\rangle = \langle f_x(x,y)a + f_y(x,y)\rangle \cdot \mathbf{u} \tag{4}$$

Remark 1. *The first vector in this dot product occurs in many contexts and is termed as* the gradient of f *and its notation is given in the following definition. (The gradient of f is also referred to as* ***grad*** *f or ∇f)*

Definition 2. If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Note. *Do not confuse with the 'Delta' symbol for changes, Δ with the gradient of f, ∇.*

Example 2. If $f(x,y) = \sin x + e^{xy}$, then

Solution.

$$\nabla f(x,y) = \langle f_x, f_y\rangle = \langle \cos x + ye^{xy}, xe^{xy}\rangle$$

and

$$\nabla f(0,1) = \langle 2,0\rangle$$

Notice that we have another notation for directional derivatives in the direction of $\mathbf{u}$, expresses as the scalar projection of the gradient vector onto $\mathbf{u}$.

$$D_u f(x,y) = \nabla f(x,y) \cdot \mathbf{u}. \tag{5}$$

Example 3. Find the directional derivative of the function $f(x,y) = x^2y^3 - 4y$ at the point $(2,-1)$ in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$.

Solution. We first compute the gradient vector at $(2,-1)$:

$$\nabla f(x,y) = 2xy^3\mathbf{i} + (3x^2y^2 - 4)\mathbf{j}$$

$$\nabla f(2,-1) = -4\mathbf{i} + 8\mathbf{j}$$

Note that $\mathbf{v}$ is not a unit vector, but since $|\mathbf{v}| = \sqrt{29}$, the unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{\sqrt{29}}\mathbf{i} + \frac{5}{\sqrt{29}}\mathbf{j}$$

Therefore, by Equation (5), we have

$$D_u f(2,-1) = \nabla f(2,-1) \cdot \mathbf{u} = (-4\mathbf{i} + 8\mathbf{j}) \cdot \left(\frac{2}{\sqrt{29}}\mathbf{i} + \frac{5}{\sqrt{29}}\mathbf{j}\right)$$

$$= \frac{-4 \cdot 2 + 8 \cdot 5}{\sqrt{29}} = \frac{32}{\sqrt{29}}$$

Three and More Variables

For functions of three variables, $D_u f(x,y,z)$ can be interpreted as the rate of change of the function in the direction of a unit vector $\mathbf{u}$.

Definition 3. The **directional derivative** of f at (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle a, b, c\rangle$ is

$$D_u f(x_0, y_0, z_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

if this limit exists.

Using vector notation, we have

$$D_u f(\mathbf{x}_0) = \lim_{h\to 0} \frac{f(\mathbf{x}_0 + h\mathbf{u}) - f(\mathbf{x}_0)}{h}$$

where $x_0 = \langle x_0, y_0\rangle$ if $n = 2$ and $x_0 = \langle x_0, y_0, z_0\rangle$ if $n = 3$.
If $f(x, y, z)$ is differentiable and $\mathbf{u} = \langle a, b, c\rangle$, then the same method that was used to prove Theorem 1 can be used here. Thus,

$$D_u f(x, y, z) = f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z)c. \tag{6}$$

For a function f of three variables, then ∇f is given by

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle$$

or simply,

$$\nabla f = \langle f_x, f_y, f_z\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Similar to functions of two variables, we can rewrite Equation (6) as follows.

$$D_u f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u} \tag{7}$$

Example 4. If $f(x, y, z) = x \sin yz$,

(a) find the gradient of f and

(b) find the directional derivative of f at $(1, 3, 0)$ in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Solution.
(a) The gradient of f is

$$\begin{aligned}\nabla f(x, y, z) &= \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle \\ &= \langle \sin yz, xz \cos yz, xy \cos yz\rangle\end{aligned}$$

(b) At $(1, 3, 0)$ we have $\nabla f(1, 3, 0) = \langle 0, 0, 3\rangle$. The unit vector in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is

$$\mathbf{u} = \frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}$$

Therefore, from Equation (7),

$$\begin{aligned}D_u f(1, 3, 0) &= \nabla f(1, 3, 0) \cdot \mathbf{u} \\ &= 3\mathbf{k} \cdot \left(\frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}\right) \\ &= 3\left(-\frac{1}{\sqrt{6}}\right) = -\sqrt{\frac{3}{2}}\end{aligned}$$