Maximizing the directional derivatives

Suppose there is a function of two variables and we consider all of its possible partial derivatives. This gives us the rates of change of the function, let say f, in all possible directions. The questions arise are: In which of these directions does the function change fastest and what is the maximum rate of change?

Theorem 1. *Suppose f is a differentiable function of two or three variables. Then, the maximum value of the directional derivative $D_u(f\boldsymbol{x})$ is $|\nabla f(\boldsymbol{x})|$ and it occurs when $\boldsymbol{u}$ has the same direction as the gradient vector $\nabla f(\boldsymbol{x})$.*

Proof. From the equations we have in the topic of directional derivatives from previous class, we have

$$D_u f = \nabla f \cdot \mathbf{u} = |\nabla f|\,|\mathbf{u}|\cos\theta = |\nabla f|\cos\theta$$

where θ is the angle between ∇f and $\mathbf{u}$. The maximum value of $\cos\theta$ is 1 and this occurs when $\theta = 0$. Therefore the maximum value of $D_u f$ is $|\nabla f|$ and it occurs when $\theta = 0$, that is, when $\mathbf{u}$ has the same direction as ∇f. □

Example 1. (a) If $f(x,y) = xe^y$, find the rate of change of f at the point $P(2,0)$ in the direction from P to $Q(\frac{1}{2}, 2)$.

(b) In what direction does f have the maximum rate of change? What is this maximum rate of change?

Solution. (a) We first compute the gradient vector:

$$\nabla f(x,y) = \langle f_x, f_y\rangle = \langle e^y, xe^y\rangle$$
$$\nabla f(2,0) = \langle 1, 2\rangle$$

The unit vector in the direction of $\vec{PQ} = \langle -1.5, 2\rangle$ is $\mathbf{u} = \left\langle -\frac{3}{5}, \frac{4}{5}\right\rangle$, so the rate of change of f in the direction from P to Q is

$$\begin{aligned} D_u f(2,0) &= \nabla f(2,0)\cdot\mathbf{u} = \langle 1,2\rangle \cdot \left\langle -\frac{3}{5}, \frac{4}{5}\right\rangle \\ &= 1(-\frac{3}{5}) + 2(\frac{4}{5}) = 1 \end{aligned}$$

(b) According to Theorem 1, f increases fastest in the direction of the gradient vector $\nabla f(2,0) = \langle 1, 2\rangle$. The maximum rate of change is

$$|\nabla f(2,0)| = |\langle 1,2\rangle| = \sqrt{5}$$

Example 2. Suppose that the temperature at a point (x, y, z) in space is given by

$$T(x, y, z) = 80/(1 + x^2 + 2y^2 + 3z^2)$$

where T is measured in degrees Celsius and x, y, z in meters. In which direction does the temperature increase fastest at the point $(1, 1, -2)$? What is the maximum rate of increase?

Solution. The gradient of T is

$$\begin{aligned}\nabla T &= \frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j} + \frac{\partial T}{\partial z}\mathbf{k} \\ &= -\frac{160x}{(1 + x^2 + 2y^2 + 3z^2)^2}\mathbf{i} - \frac{320y}{(1 + x^2 + 2y^2 + 3z^2)^2}\mathbf{j} - \frac{480z}{(1 + x^2 + 2y^2 + 3z^2)^2)}\mathbf{k} \\ &= \frac{160}{(1 + x^2 + 2y^2 + 3z^2)^2}(-x\mathbf{i} - 2y\mathbf{j} - 3z\mathbf{k})\end{aligned}$$

At the point $(1, 1, -2)$ the gradient vector is

$$\nabla T(1, 1, -2) = \frac{160}{256}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}) = \frac{5}{8}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})$$

By Theorem 1, the temperature increases fastest in the direction of the gradient vector $\nabla T(1, 1, -2) = \frac{5}{8}(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})$ or, equivalently, in the direction of $-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$ or the unit vector $(-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})/\sqrt{41}$. The maximum rate of increase is the length of the gradient vector:

$$|\nabla T(1, 1, -2)| = \frac{5}{8}\,|-\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}| = \frac{5}{8}\sqrt{41}$$

Therefore the maximum rate of increase of temperature is $\frac{5}{8}\sqrt{41} \approx 4\,°\mathrm{C}/m$

Tangent planes to level surfaces

Suppose S is a surface with equation $F(x, y, z) = k$, that is, it is a level surface of a function F of three variables, and let $P(x_0, y_0, z_0)$ be a point on S. Let C be any curve that lies on the surface S and passes through the point P. Recall that the curve can be described by a continuous vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$. Let t_0 be the parameter value corresponding to P; that is, $\mathbf{r}(t_0) = \langle x_0, y_0, z_0\rangle$. Since C lies on S, any point $x(t), y(t), z(t)$ must satisfy the equation of S, that is,

$$F(x(t), y(t), z(t)) = k$$

If x, y, and z are differentiable functions of t and F is also differentiable, then we can use the Chain Rule to differentiate the previous equation.

$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0 \qquad (1)$$

But since $\nabla F = \langle F_x, F_y, F_z \rangle$ and $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$, Equation (1) can be written in terms of a dot product as

$$\nabla F \cdot \mathbf{r}'(t) = 0 \tag{2}$$

In particular, when $t = t_0$, we have $\mathbf{r}(t_0) = \langle x_0, y_0, z_0 \rangle$. So,

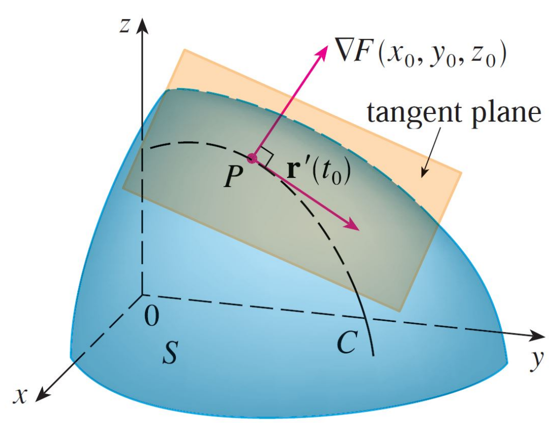
$$\nabla F(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) = 0 \tag{3}$$

Equation (3) says that the gradient vector at P, $\nabla F(x_0, y_0, z_0)$, is perpendicular to the tangent vector $\mathbf{r}'(t_0)$ to any curve C on S that passes through P. If $\nabla F(x_0, y_0, z_0) \neq 0$, it is therefore natural to define the **tangent plane to the level surface** $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$ as the plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$. Using the standard equation of a plane, we can write the equation of this tangent plane as

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0 \tag{4}$$

The **normal line** to S at P is the line passing through P and perpendicular to the tangent plane. The direction of the normal line is therefore given by the gradient vector $\nabla F(x_0, y_0, z_0)$ and so,

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)} \tag{5}$$



In the special case in which the equation of a surface S is of the form $z = f(x, y)$ (that is, S is the graph of a function of two variables), we can rewrite the equation as

$$F(x, y, z) = f(x, y) - z = 0$$

and regard S as a level surface (with $k = 0$) of F. Then

$$F_x(x_0, y_0, z_0) = f_x(x_0, y_0)$$
$$F_y(x_0, y_0, z_0) = f_y(x_0, y_0)$$
$$F_z(x_0, y_0, z_0) = -1$$

So, Equation (4) becomes

$$f_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$$

Example 3. Find the equations of the tangent plane and normal line at the point $(-2, 1, -3)$ to the ellipsoid

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$$

Solution. The ellipsoid is the level surface (with $k = 3$) of the function

$$F(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$$

Therefore we have

$$F_x(x, y, z) = \frac{x}{2} \quad F_y(x, y, z) = 2y \quad F_z(x, y, z) = \frac{2z}{9}$$
$$F_x(-2, 1, -3) = -1 \quad F_y(-2, 1, -3) = 2 \quad F_z(-2, 1, -3) = -\frac{2}{3}$$

Then Equation 4 gives the equation of the tangent plane at $(-2, 1 - 3)$ as

$$-1(x + 2) + 2(y - 1) - \frac{2}{3}(z + 3) = 0$$

which simplifies to $3x - 6y + 2z + 18 = 0$.

By Equation 5, symmetric equations of the normal line are

$$\frac{x + 2}{-1} = \frac{y - 1}{2} = \frac{z + 3}{-\frac{2}{3}}$$