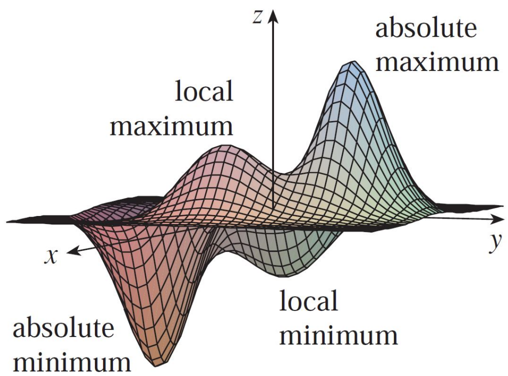
Maximum and Minimum Values

Similar to functions of single variable, we have the concept of minimum and maximum for function of several variables. For example, a hill can be plotted into function of two variables. Notice that, we can always locate the high and low point of the hill, where in Calculus, this represent the minimum and maximum points. Consider the following figure for function of two variables.



There are two points (a, b) where f has a local maximum, that is, where $f(a, b)$ is larger than nearby values of $f(x, y)$. The larger of these two values is the **absolute maximum**. Likewise, f has two local minima, where $f(a, b)$ is smaller than nearby values. The smaller of these two values is the **absolute minimum**.

Definition 1. A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leqslant f(a, b)$ when (x, y) is near (a, b). [This means that $f(x, y) \leqslant f(a, b)$ for all points (x, y) in some disk with center (a, b) .] The number $f(a, b)$ is called a **local maximum value**. If $f(x, y) \geqslant f(a, b)$ when (x, y) is near (a, b), then f has a **local minimum** at (a, b) and $f(a, b)$ is a **local minimum value**.

Note. *If Definition 1 holds, then f has an absolute maximum (or absolute minimum) at (a, b).*

Theorem 1. *If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.*

Proof. Let $g(x) = f(x, b)$. If f has a local maximum (or minimum) at (a, b), then g has a local maximum (or minimum) at a, so $g'(a) = 0$ by Fermat's Theorem (see Theorem 4.1.4 in book). But $g'(a) = f_x(a, b)$ and so $f_x(a, b) = 0$. Similarly, by applying Fermat's Theorem to the function $G(y) = f(a, y)$, we obtain $f_y(a, b) = 0$. □

If we put $f_x(a, b) = 0$ and $f_y(a, b) = 0$ in the equation of a tangent plane, we get $z = z_0$. Thus the geometric interpretation of Theorem 2 is that if the graph of f has a tangent plane at a local maximum or minimum, then the tangent plane must be horizontal.
A point is called a **critical point** (or stationary point) of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist. Theorem 2 says that if f has a local maximum or minimum at (a, b), then (a, b) is a critical point of f. However, as in MAT101, not all critical points give rise to maxima or minima. At a critical point, a function could have a local maximum or a local minimum or neither.

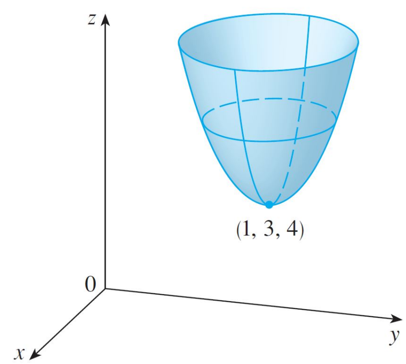
Example 1. Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14$. Then

$$f_x(x, y) = 2x - 2 \qquad\qquad f_y(x, y) = 2y - 6$$

Solution. These partial derivatives are equal to 0 when $x = 1$ and $y = 3$, so the only critical point is $(1, 3)$. By completing the square, we find that

$$f(x, y) = 4 + (x - 1)^2 + (y - 3)^2$$

Since $(x - 1)^2 \geqslant 0$ and $(y - 3)^2 \geqslant 0$, we have $f(x, y) \geqslant 4$ for all values of x and y. Therefore $f(1, 3) = 4$ is a local minimum, and in fact it is the absolute minimum of f. This can be confirmed geometrically from the graph of f which is the elliptic paraboloid with vertex $(1, 3, 4)$ shown in figure below.



Example 2. Find the extreme values of $f(x, y) = y^2 - x^2$.

Solution. Since $f_x = -2x$ and $f_y = 2y$, the only critical point is $(0, 0)$. Notice that for points on the x-axis we have $y = 0$, so $f(x, y) = -x^2 < 0$ (if $x \neq 0$). However, for points on the y-axis we have $x = 0$, so $f(x, y) = y^2 > 0$ (if $y \neq 0$). Thus every disk with center $(0, 0)$ contains points where f takes positive values as well as points where f takes negative values. Therefore $f(0, 0) = 0$ can't be an extreme value for f, so f has no extreme value.

The following theorem describe one of the most important test in this topic.

Theorem 2. *(*Second Derivative Test*) Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$* (that is, (a, b) is a critical point of f)*. Let*

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

(1) *If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.*
(2) *If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.*
(3) *If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.*

Proof. We compute the second-order directional derivative of f in the direction of $\mathbf{u} = \langle h, k \rangle$. The first order derivative is given by

$$D_u f = f_x h + f_y k$$

If we perform second differentiation on the same function,

$$
\begin{aligned}
D_u^2 f = D_u(d_u f) &= \frac{\partial}{\partial x}(D_u f)h + \frac{\partial}{\partial y}(D_u f)k \\
&= (f_{xx}h + f_{yx}k)h + (f_{xy}h + f_{yy}k)k \\
&= f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2
\end{aligned}
$$

Completing the square, we obtain

$$
D_u^2 f = f_{xx}\left(h + \frac{f_{xy}}{f_x x}k\right)^2 + \frac{k^2}{f_{xx}}(f_{xx}f_{yy} - f_{xy}^2)
$$

We are given that $f_{xx}(a,b) > 0$ and $D(a,b) > 0$. But $f_x xx$ and D are continuous functions, so there is a disk B with center (a,b) and radius $\delta > 0$ such that $f_{xx}(x,y) > 0$ and $D(x,y) > 0$ whenever (x,y) is in B. Therefore, by looking at Equation (), we see that $D_u^2 f(x,y) > 0$ whenever (x,y) is in B. This means that if C is the curve obtained by intersecting the graph of f with the vertical plane through $P(a,b,f(a,b))$ in the direction of $\mathbf{u}$, then C is concave upward on an interval of length 2δ. This is true in the direction of every vector $\mathbf{u}$, so if we restrict (x,y) to lie in B, the graph of f lies above its horizontal tangent plane at P. Thus $f(x,y) \geq f(a,b)$ whenever (x,y) is in B. This shows that $f(a,b)$ is a local minimum. □

Note. *In Case 3 of Theorem 2, the point is called **saddle point**.*

Example 3. Find the local maximum and minimum values of the saddle points of $f(x,y) = x^4 + y^4 - 4xy + 1$.

Solution. We first locate the critical points:

$$
f_x = 4x^3 - 4y \qquad f_y = 4y^3 - 4x
$$

To solve these equations, we substitute $y = x^3$ from the first equation into the second one. This gives

$$
0 = x^9 - x = x(x^8 - 1) = x(x^4 - 1)(x^4 + 1) = x(x^2 - 1)(x^2 + 1)(x^4 + 1)
$$

so there are three real roots: $x = 0, 1, -1$. The three critical points are $(0,0)$, $(1,1)$ and $(-1,-1)$.

Next we calculate the second partial derivatives and $D(x,y)$:

$$
\begin{aligned}
f_{xx} &= 12x^2 \\
f_{xy} &= -4 \\
f_{yy} &= 12y^2 \\
D(x,y) &= f_{xx}f_{yy} - (f_{xy})^2 = 144x^2y^2 - 16
\end{aligned}
$$

Since $D(0,0) = -16 < 0$, by using Second Derivative Test, we conclude that the origin is a **saddle point**. We also obtain that $D(1,1) = 128 > 0$ and $f_{xx}(1,1) = 12 > 0$, by using Second Derivative Test, we see that $f(1,1) = -1$ is a **local minimum**. Similarly $D(-1,-1) = 128 > 0$ and $f_{xx}(-1,-1) = 12 > 0$,so $f(-1,-1) = -1$ is also a **local minimum**.

Example 4. Find the shortest distance from the point $(1,0,-2)$ to the plane $x + 2y + z = 4$.

Solution. The distance from any point (x,y,z) to the point $(1,0,-2)$ is

$$
d = \sqrt{(x-1)^2 + y^2 + (z+2)^2}
$$

but if (x, y, z) lies on the plane $x + 2y + z = 4$, then $z = 4 - x - 2y$ and so we have $d = \sqrt{(x-1)^2 + y^2 + (6 - x - 2y)^2}$. We can minimize d by minimizing the simpler expression

$$d^2 = f(x, y) = (x-1)^2 + y^2 + (6 - x - 2y)^2$$

Then,

$$\begin{aligned} f_x &= 2(x-1) - 2(6 - x - 2y) = 4x + 4y - 14 = 0 \\ f_y &= 2y - 4(6 - x - 2y) = 4x + 10y - 24 = 0 \end{aligned}$$

we find that the only critical point is $\left(\frac{11}{6}, \frac{5}{3}\right)$. Since $f_{xx} = 4$, $f_{xy} = 4$, and $f_{yy} = 10$, we have $D(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = 24 > 0$ and $f_{xx} > 0$, so by the Second Derivatives Test, f has a local minimum at $\left(\frac{11}{6}, \frac{5}{3}\right)$. Intuitively, we can see that this local minimum is actually an absolute minimum because there must be a point on the given plane that is closest to $(1, 0, -2)$. If $x = \frac{11}{6}$ and $y = \frac{5}{3}$, then

$$d = \sqrt{(x-1)^2 + y^2 + (6 - x - 2y)^2} = \sqrt{\left(\frac{5}{6}\right)^2 + \left(\frac{5}{3}\right)^2 + \left(\frac{5}{6}\right)^2} = \frac{5}{6}\sqrt{6}$$

The shortest distance from $(1, 0, -2)$ to the plane $x + 2y + z = 4$ is $\dfrac{5}{6}\sqrt{6}$.

Absolute Maximum and Minimum Values

For function of single variable that is continuous on $[a, b]$, to find absolute maximum and minimum values, we also need to consider the point at the boundary of the domain. This is similar to function of two variables. Just as a closed interval contains its endpoints, a closed set in $\mathbb{R}^2$ is one that contains all its boundary points. [A boundary point of D is a point (a, b) such that every disk with center (a,b) contains points in D and also points not in D.] For instance, the disk

$$D = \{(x, y) | x^2 + y^2 \leq 1\}$$

which consists of all points on and inside the circle $x^2 + y^2 = 1$, is a closed set because it contains all of its boundary points (which are the points on the circle $x^2 + y^2 = 1$). But if even one point on the boundary curve were omitted, the set would not be closed.

A bounded set in $\mathbb{R}^2$ is one that is contained within some disk. In other words, it is finite in extent. Then, in terms of closed and bounded sets, we can state the following counterpart of the Extreme Value Theorem in two dimensions.

Theorem 3. *(*Extreme Value Theorem for Functions of Two Variables*) If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points and (x_1, y_1) and (x_2, y_2) in D.*

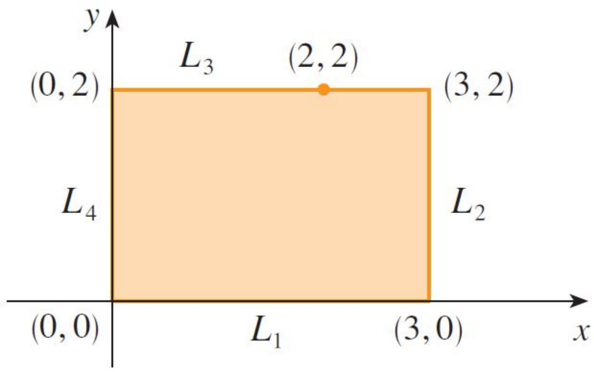
To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

(1) Find the values of f at the critical points of f in D.

(2) Find the extreme values of f on the boundary of D.

(3) The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example 5. Find the absolute maximum and minimum values of the function $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x, y)|0 \leqslant x \leqslant 3, 0 \leqslant y \leqslant 2\}$.



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Solution. Since f is a polynomial, it is continuous on the closed, bounded rectangle D, so Theorem 3 tells us there is both an absolute maximum and an absolute minimum. We find that the critical points occur when

$$f_x = 2x - 2y = 0 \qquad\qquad f_y = -2x + 2 = 0$$

so the only critical point is $(1, 1)$, and the value of f there is $f(1, 1) = 1$.

Then, we look at the values of f on the boundary of D, which consists of the four line segments L_1, L_2, L_3, L_4 shown in figure above. On L_1, we have $y = 0$ and

$$f(x, 0) = x^2 \qquad 0 \leqslant x \leqslant 3$$

This is an increasing function of x, so its minimum value is $f(0, 0) = 0$ and its maximum value is $f(3, 0) = 9$. On L_2 we have $x = 3$ and

$$f(3, y) = 9 - 4y \qquad 0 \leqslant y \leqslant 2$$

This is a decreasing function of y, so its maximum value is $f(3, 0) = 9$ and its minimum value is $f(3, 2) = 1$. On L_3 we have $y = 2$ and

$$f(x, 2) = x^2 - 4x + 4 \qquad 0 \leqslant x \leqslant 3$$

By observing that $f(x, 2) = (x-2)^2$, we see that the minimum value of this function is $f(2, 2) = 0$ and the maximum value is $f(0, 2) = 4$. Finally, on L_4 we have $x = 0$ and

$$f(0, y) = 2y \qquad 0 \leqslant y \leqslant 2$$

with maximum value $f(0, 2) = 4$ and minimum value $f(0, 0) = 0$. Thus, on the boundary, the minimum value of f is 0 and the maximum is 9.

Then, we compare these values with the value $f(1, 1) = 1$ at the critical point and conclude that the absolute maximum value of f on D is $f(3, 0) = 9$ and the absolute minimum value is $f(0, 0) = f(2, 2) = 0$.