Volumes and Double Integrals

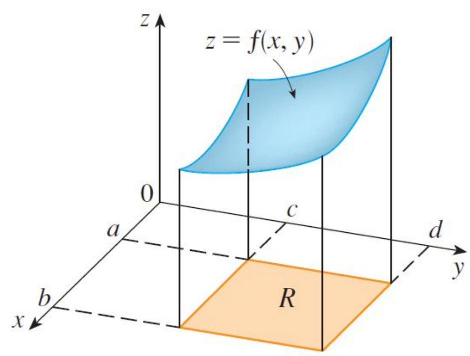
Consider a function f of two variables defined on a closed rectangle

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$$

and we first suppose that $f(x, y) \geqslant 0$. The graph of f is a surface with equation $z = f(x, y)$. Let S be the solid that lies above R and under the graph of f, that is,

$$S = \{(x, y, z) \in \mathbb{R}^3 | 0 \leqslant z \leqslant f(x, y), (x, y) \in R\}$$

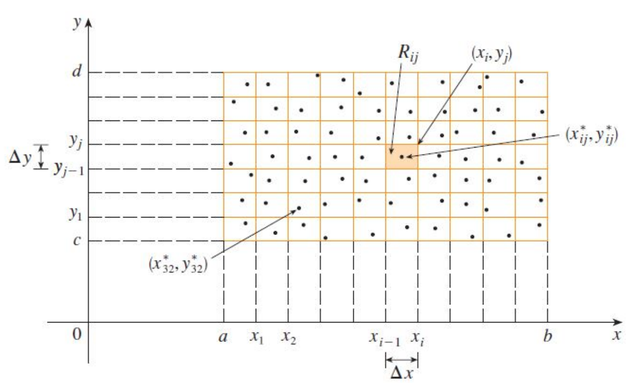
(As in the figure below) Our goal is to find the volume of S.



The first step is to divide the rectangle R into subrectangles. We accomplish this by dividing the interval $[a, b]$ into m subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b - a)/m$ and dividing $[c, d]$ into n subintervals $[y_{j-1}, y_j]$ of equal width $\Delta y = (d - c)/n$ By drawing lines parallel to the coordinate axes through the endpoints of these subintervals, as in figure below, we form the subrectangles

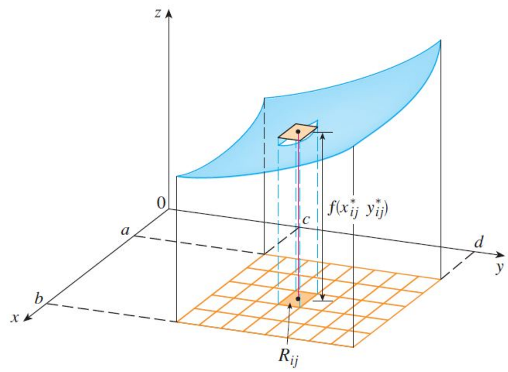
$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x, y) | x_{i-1} \leqslant x \leqslant x_i, y_{j-1} \leqslant y \leqslant y_j\}$$

each with area $\Delta A = \Delta x \Delta y$.



If we choose a **sample point** (x_{ij}^*, y_{ij}^*) in each R_{ij}, then we can approximate the part of S that lies above each R_{ij} by a thin rectangular box (or column) with base R_{ij} and height $f(x_{ij}^*, y_{ij}^*)$ as shown in figure below. The volume of this box is the height of the box times the area of the base rectangle:

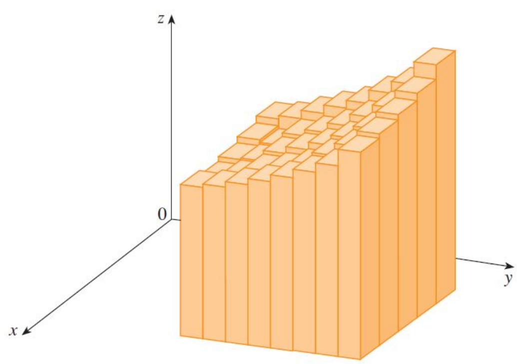
$$f(x^*_{ij}, y^*_{ij})\Delta A$$



If we follow this procedure for all the rectangles and add the volumes of the corresponding boxes, we get an approximation to the total volume of S:

$$V \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij})\Delta A \tag{1}$$

(See Figure below) This double sum means that for each subrectangle we evaluate f at the chosen point and multiply by the area of the subrectangle, and then we add the results.



Our intuition tells us that the approximation given in Equation (1) becomes better as m and n become larger and so we would expect that

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij})\Delta A \tag{2}$$

We use the expression in Equation (2) to define the **volume** of the solid S that lies under the graph of f and above the rectangle R.

Limits of the type that appear in Equation (2) occur frequently, not just in finding volumes but in a variety of other situations as wellas we will see in incoming lecture, even f when is not a positive function. So we introduce the following definition.

Definition 1. The **double integral** of f over the rectangle R is

$$\int\int_R f(x, y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij})\Delta A$$

if this limit exists.

The precise meaning of the limit in Definition 1 is that for every number $\epsilon > 0$ there is an integer N such that

$$\left| \int \int_R f(x,y)dA - \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A \right| < \epsilon$$

for all integers m and n greater than N and for any choice of sample points (x_{ij}^*, y_{ij}^*) in R_{ij}.

A function f is called **integrable** if the limit in Definition 1 exists. It is known that all continuous functions are integrable. In fact, the double integral of f exists provided that f is not too discontinuous". In particular, if f is bounded [that is, there is a constant M such that $|f(x,y)| \leqslant M$ for all (x,y) in R], and f is continuous there, except on a finite number of smooth curves, then f is integrable over R.

The sample point (x_{ij}^*, y_{ij}^*) can be chosen to be any point in the subrectangle R_{ij} but if we choose it to be the upper right-hand corner of R_{ij} [namely (x_i, y_j), see the second figure in this lecture note], then the expression for double integral looks simpler:

$$\int \int_R f(x,y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\Delta A$$

By comparing Equation (2) and equation in Definition 1, we see that a volume can be written as a double integral:

If $f(x,y) \geqslant 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x,y)$ is

$$V = \int \int_R f(x,y)dA$$

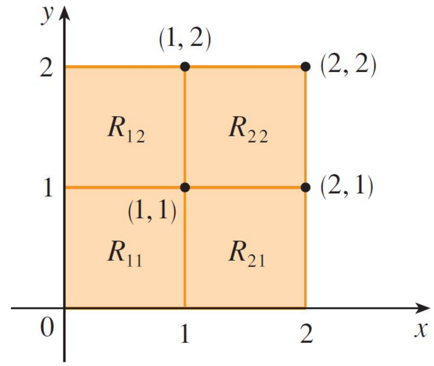
The sum in Definition (1),

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

is called a **double Riemann sum** and is used as an approximation to the value of the double integral. If f happens to be a *positive* function, then the double Riemann sum represents the sum of volumes of columns, and is an approximation to the volume under the graph of f and above the rectangle R.

Example 1. Estimate the volume of the solid that lies above the square $R = [0,2] \times [0,2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into four equal squares and choose the sample point to be the upper right corner of each square R_{ij}. Sketch the solid and the approximating rectangular boxes.

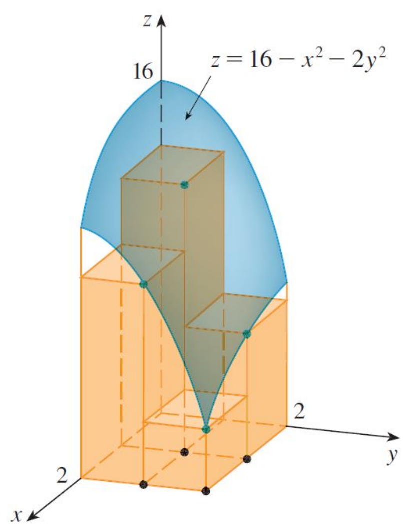
Solution. The squares are shown in figure below.



The paraboloid is the graph of $f(x, y) = 16 - x^2 - 2y^2$ and the area of each square is 1. Approximating the volume by the Riemann sum with $m = n = 2$, we have

$$\begin{aligned} V &\approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_i, y_j) \Delta A \\ &= f(1, 1)\Delta A + f(1, 2)\Delta A + f(2, 1)\Delta A + f(2, 2)\Delta A \\ &= 13(1) + 7(1) + 10(1) + 4(1) = 34 \end{aligned}$$

This is the volume of the approximating rectangular boxes shown in the following figure.

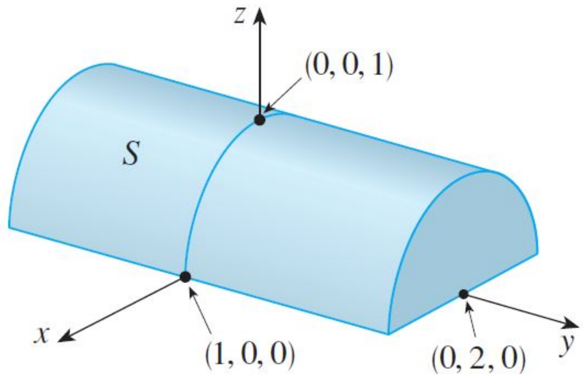


Example 2. If $R = \{(x, y| -1 \leqslant x \leqslant 1, -2 \leqslant y \leqslant 2)\}$, evaluate the integral

$$\int\int_R \sqrt{1 - x^2} dA$$

Solution. It would be very difficult to evaluate this integral directly from Definition (1) but, because $\sqrt{1 - x^2} \geqslant 0$, we can compute the integral by interpreting it as a volume. If $z = \sqrt{1 - x^2}$, then $x^2 + z^2 = 1$ and $z \geqslant 0$, so the given double integral represents the volume of the solid S that lies below the circular cylinder $x^2 + z^2 = 1$ and above the rectangle R. The volume of S is the area of a semicircle with radius 1 times the length of the cylinder. Thus

$$\int\int_R \sqrt{1 - x^2} dA = \frac{1}{2}\pi(1)^2 \times 4 = 2\pi$$



Midpoint Rule

Recall the methods we used in MAT101 such as Midpoint Rule, Trapezoidal Rule and Simpson's Rule. All of them have counterparts in double integral. In this topic, we will consider Midpoint Rule for double integrals. This means that we use a double Riemann sum to approximate the double integral, where the sample point (x_{ij}^*, y_{ij}^*) in R_{ij} is chosen to be the center $(\overline{x}_i, \overline{y}_j)$ of R_{ij}. In other words, $\overline{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and is $\overline{y}_i$ is the midpoint of $[y_{i-1}, y_i]$. Then,

$$\int\int_R f(x,y)dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\overline{x}_i, \overline{y}_j)\Delta A.$$

Example 3. Use the Midpoint Rule with $m = n = 2$ to estimate the value of the integral $\int\int_R (x - 3y^2)dA$, where $R = \{(x, y)|0 \leqslant x \leqslant 2, 1 \leqslant y \leqslant 2\}$.

Solution. In using the Midpoint Rule with $m = n = 2$, we evaluate $f(x, y) = x - 3y^2$ at the centers of the four subrectangles shown in Figure 10. So $\overline{x}_1 = \frac{1}{2}$, $\overline{x}_2 = \frac{3}{2}$, $\overline{y}_1 \frac{5}{4}$, and $\overline{y}_2 = \frac{7}{4}$. The area of each subrectangle is $\Delta A = \frac{1}{2}$. Thus

$$\begin{aligned}
\int\int_R (x - 3y^2)dA &\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(\overline{x}_i, \overline{y}_j)\Delta A \\
&= f(\overline{x}_1, \overline{y}_1)\Delta A + f(\overline{x}_1, \overline{y}_2)\Delta A + f(\overline{x}_2, \overline{y}_1)\Delta A + f(\overline{x}_2, \overline{y}_2)\Delta A \\
&= f(\frac{1}{2}, \frac{5}{4})\Delta A + f(\frac{1}{2}, \frac{7}{4})\Delta A + f(\frac{3}{2}, \frac{5}{4})\Delta A + f(\frac{3}{2}, \frac{7}{4})\Delta A \\
&= (-\frac{67}{16})\frac{1}{2} + (-\frac{139}{16})\frac{1}{2} + (-\frac{51}{16})\frac{1}{2} + (-\frac{123}{16})\frac{1}{2} \\
&= -\frac{95}{8} = -11.875
\end{aligned}$$

Thus we have

$$\int\int_R (x - 3y^2)dA \approx -11.875$$

Average Value

Recall that the average value of a function f of one variable defined on an interval $[a, b]$ is given as

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

We can extend this for double integrals defined on a rectangle R to be

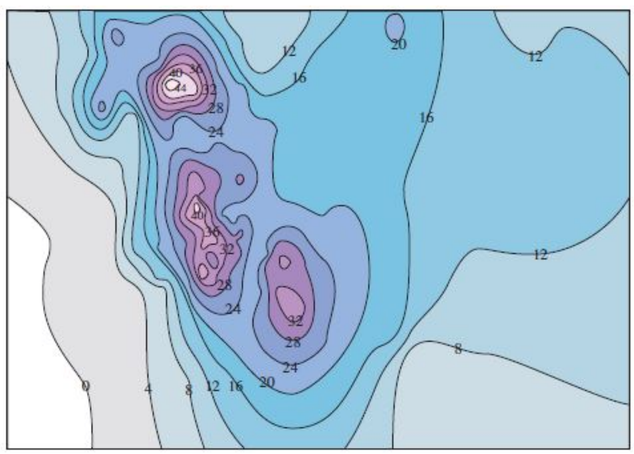
$$f_{ave} = \frac{1}{A(R)} \int\int_R f(x, y)\, dA$$

where $A(R)$ is the area of R. If $f(x, y) \geq 0$, the equation

$$A(R) \times f_{ave} = \int\int_R f(x, y)\, dA$$

says that the box with base R and f_{ave} height has the same volume as the solid that lies under the graph of f.

Example 4. The contour map in figure below shows the snowfall, in inches, that fell on the state of Colorado on December 20 and 21, 2006. (The state is in the shape of a rectangle that measures 388 miles west to east and 276 miles south to north.) Use the contour map to estimate the average snowfall for the entire state of Colorado on those days.

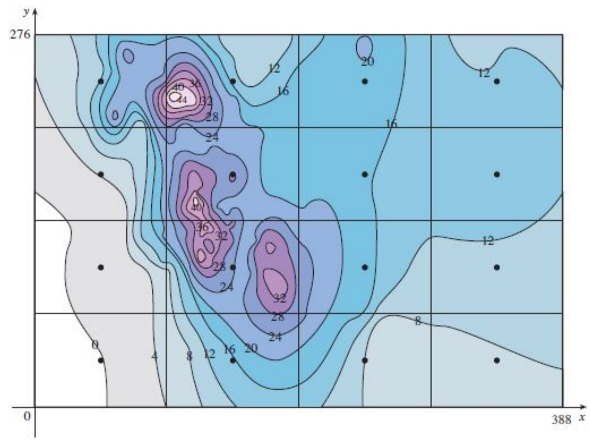


Solution. Let's place the origin at the southwest corner of the state. Then $0 \leq x \leq 388$, $0 \leq y \leq 276$, and $f(x, y)$ is the snowfall, in inches, at the location x miles to the east and y miles to the north of the origin. If R is the rectangle that represents Colorado, then the average snowfall for the state on December 20-21 was

$$f_{ave} = \frac{1}{A(R)} \int\int_R f(x, y)\, dA$$

where $A(R) = 388.276$. To estimate the value of this double integral, let's use Midpoint Rule with $m = n = 4$.In other words, we divide R into 16 subrectangles of equal size, as in figure below. The area of each subrectangle is

$$\Delta A = \frac{1}{160}(388)(276) = 6693 \text{ miles}^2$$



Using the contour map to estimate the value of f at the center of each subrectangle, we get

$$
\begin{aligned}
\int\int_R f(x,y)\,dA &\approx \sum_{i=1}^{4}\sum_{j=1}^{4} f(\overline{x}_i,\overline{x}_j)\,\Delta A \\
&\approx \Delta A[0+15+8+7+2+25+18.5+11+4.5+28+17+13.5 \\
&\qquad +12+15+17.5+13] \\
&= (6693)(207)
\end{aligned}
$$

Therefore,

$$f_{ave} = \frac{(6693)(207)}{(388)(276)}$$

We may conclude that on December 20-21, 2006, Colorado received an average of approximately 13 inches of snow.

Properties of Double Integrals

We list here three properties of double integrals that can be proved in the same manner as in MAT101. We assume that all of the integrals exist. The first two properties below are referred to as the linearity of the integral.

$$\int\int_R [f(x,y)+g(x,y)]\,dA = \int\int_R f(x,y)\,dA + \int\int_R g(x,y)\,dA$$

$$\int\int_R cf(x,y)\,dA = c\int\int_R f(x,y)\,dA \qquad \text{where } c \text{ is a constant.}$$

If $f(x,y) \geq g(x,y)$ for all (x,y) in R, then

$$\int\int_R f(x,y)\,dA \geq \int\int_R g(x,y)\,dA.$$