Iterated Integrals

The evaluation of double integrals from first principles (from the last lecture) is even more difficult, but in this lecture, we see how to express a double integral as an iterated integral, which can then be evaluated by calculating two single integrals.

Suppose that f is a function of two variables that is integrable on the rectangle $R = [a, b] \times [c, d]$. We use the notation $\int_c^d f(x, y)\, dy$ to mean that x is held fixed and $f(x, y)$ is integrated with respect to y from $y = c$ to $y = d$. This procedure is called *partial integration* with respect to y. (Notice its similarity to partial differentiation.) Now $\int_c^d f(x, y) dy$ is a number that depends on the value of x, so it defines a function of x:

$$A(x) = \int_c^d f(x, y)\, dy$$

If we now integrate the function A with respect to x from $x = a$ to $x = b$, we get

$$\int_a^b A(x)\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx \tag{1}$$

The integral on the right side of Equation 1 is called an **iterated integral**. Usually the brackets are omitted. Thus

$$\int_a^b \int_c^d f(x, y)\, dy\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx \tag{2}$$

means that we first integrate with respect to y from c to d and then with respect to x from b to b. Similarly, the iterated integral

$$\int_c^d \int_a^b f(x, y)\, dx\, dy = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy \tag{3}$$

means that we first integrate with respect to x (holding y fixed) from $x = a$ to $x = b$ and then we integrate the resulting function of y with respect to y from $y = c$ to $y = d$. Notice that in both Equations (2) and (3), we work from the inside out.

Example 1. Evaluate the iterated integrals.

(1) $\int_0^3 \int_1^2 x^2 y\, dy dx$
(2) $\int_1^2 \int_0^3 x^2 y\, dx dy$

Solution.

(1) Regarding x as a constant, we obtain

$$\int_1^2 x^2 y\, dy = \left[x^2 \frac{y^2}{2} \right]_{y=1}^{y=2} = x^2 \left(\frac{2^2}{2} \right) - x^2 \left(\frac{1^2}{2} \right) = \frac{3}{2} x^2$$

Thus the function A in the preceding discussion is given by $A(x) = \frac{3}{2}x^2$ in this example. We now integrate this function of x from 0 to 3:

$$\begin{aligned}\int_0^3 \int_1^2 x^2 y \, dy dx &= \int_0^3 \left[\int_1^2 x^2 y \, dy \right] dx \\ &= \int_0^3 \frac{3}{2} x^2 \, dx = \left[\frac{x^3}{2} \right]_0^3 = \frac{27}{2}\end{aligned}$$

(2) Here we first integrate with respect to x:

$$\begin{aligned}\int_1^2 \int_0^3 x^2 y \, dx dy &= \int_1^2 \left[\int_0^3 x^2 y \, dx \right] dy = \int_1^2 \left[\frac{x^3}{3} y \right]_{x=0}^{x=3} dy \\ &= \int_1^2 9y \, dy = 9 \left[\frac{y^2}{2} \right]_1^2 = \frac{27}{2}\end{aligned}$$

Notice from the example, we get the same value no matter we integrate with respect to x first or with respect to y first. This is in accord with Fubini's Theorem.

Theorem 1. Fubini's Theorem *If f is continuous on the rectangle*

$$R = \{(x, y) | a \leqslant x \leqslant b, c \leqslant y \leqslant d\},$$

then

$$\int\int_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy$$

More generally, this is true if we assume that f is bounded on R, is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

The proof of Fubini's Theorem is excluded from this lecture note since it involves knowledge from analysis. But we shall give intuition on why the theorem is true for case $f(x, y) \geq 0$. Recall that if f is positive, then we can interpret the double integral $\int\int_R f(x, y) \, dA$ as the volume V of the solid S that lies above R and under the surface $z = f(x, y)$. But we have another formula that we used for volume in MAT101, namely,

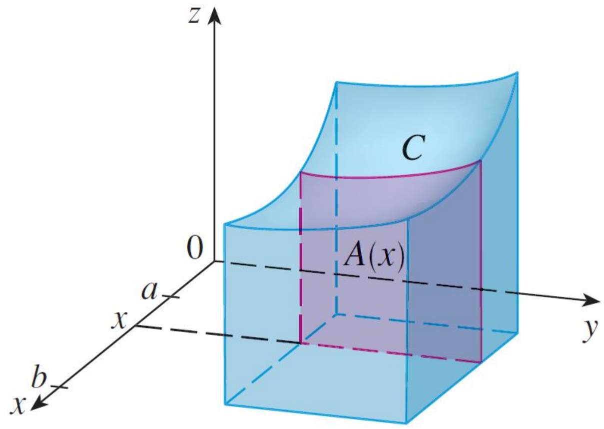
$$V = \int_a^b A(x) \, dx$$

where $A(x)$ is the area of a cross-section of in the plane through x perpendicular to the x-axis. From figure below, we can see that $A(x)$ is the area under the curve C whose equation is $z = f(x, y)$, where x is held constant and $c \leq y \leq d$. Therefore

$$A(x) = \int_c^d f(x, y) dy$$

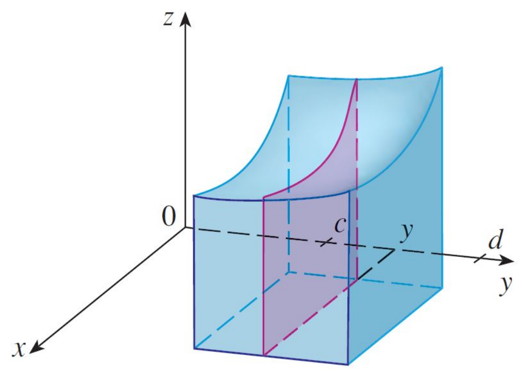
and

$$\int\int_R f(x, y) \, dA = V = \int_a^b A(x) \, dx = \int_a^b \int_c^d f(x, y) \, dy \, dx$$



Using similar argument, using cross-sections perpendicular to the y-axis as in figure below, we have

$$\int\int_R f(x,y)\,dA = V = \int_c^d \int_a^b f(x,y)\,dx\,dy$$



Example 2. Evaluate $\int\int_R y\sin(xy)\,dA$, where $R = [1,2]\times[0,\pi]$.

Solution.

(1) If we first integrate with respect to x, we get

$$\begin{aligned}\int\int_R y\sin(xy)\,dA &= \int_0^\pi \int_1^2 y\sin(xy)\,dxdy = \int_0^\pi [-\cos(xy)]_{x=1}^{x=2} \\ &= \int_0^\pi (-\cos 2y + \cos y)dy \\ &= -\left[\frac{1}{2}\sin 2y + \sin y\right]_0^\pi = 0\end{aligned}$$

(2) If we reverse the order of integration, we get

$$\int\int_R y\sin(xy)\,dA = \int_1^2 \int_0^\pi y\sin(xy)\,dy\,dx$$

To evaluate the inner integral, we use integration by parts with

$$u = y \qquad dv = \sin(xy)\,dy$$
$$du = dy \qquad v = -\frac{\cos(xy)}{x}$$

and so

$$\begin{aligned}\int_0^\pi y\sin(xy)\,dy &= -\left[\frac{y\cos(xy)}{x}\right]_{y=0}^{y=\pi} + \frac{1}{x}\int_0^\pi \cos(xy)\,dy \\ &= -\frac{\pi\cos\pi x}{x} + \frac{1}{x^2}\left[\sin(xy)\right]_{y=0}^{y=\pi} \\ &= -\frac{\pi\cos\pi x}{x} + \frac{\sin\pi x}{x^2}\end{aligned}$$

If we now integrate the first term by parts with $u = -1/x$ and $dv = \pi\cos\pi x\,dx$, we get $du = \dfrac{dx}{x^2}$, $v = \sin\pi x$, and

$$\int\left(-\frac{\pi\cos\pi x}{x}\right)dx = -\frac{\sin\pi x}{x} - \int\frac{\sin\pi x}{x^2}dx$$

Therefore

$$\int\left(-\frac{\pi\cos\pi x}{x} + \frac{\sin\pi x}{x^2}\right)dx = -\frac{\sin\pi x}{x}$$

and so

$$\begin{aligned}\int_1^2\int_0^\pi y\sin(xy)\,dydx &= \left[-\frac{\sin\pi x}{x}\right]_1^2 \\ &= -\frac{\sin 2\pi}{2} + \sin\pi = 0\end{aligned}$$

In the special case where $f(x,y)$ can be factored as the product of a function of x only and a function of y only, the double integral of f can be written in a particularly simple form. To be specific, suppose that $f(x,y) = g(x)h(y)$ and $R = [a,b]\times[c,d]$. Then Fubini's Theorem gives

$$\int\int_R f(x,y)\,dA = \int_c^d\int_a^b g(x)h(y)\,dxdy = \int_c^d\left[\int_a^b g(x)h(y)\,dx\right]dy$$

In the inner integral, y is a constant, so $h(y)$ is a constant and we can write

$$\int_c^d\left[\int_a^b g(x)h(y)\,dx\right]dy = \int_c^d\left[h(y)\int_a^b g(x)\,dx\right]dy = \int_a^b g(x)\,dx\int_c^d h(y)\,dy$$

since $\int_a^b g(x)\,dx$ is a constant. Therefore, in this case, the double integral of f can be written as the product of two single integrals:

$$\boxed{\int\int_R g(x)h(y)\,dA = \int_a^b g(x)\,dx\int_c^d h(y)dy \qquad \text{where } R = [a,b]\times[c,d]} \tag{4}$$

Example 3. Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes.

Solution. We first observe that S is the solid that lies under the surface $z = 16 - x^2 - 2y^2$ and above the square $R = [0,2]\times[0,2]$. We are now in a position to evaluate the double integral using Fubini's Theorem. Therefore

$$\begin{aligned}
V &= \int\int_R (16 - x^2 - 2y^2)\, dA = \int_0^2 \int_0^2 (16 - x^2 - 2y^2)\, dx\, dy \\
&= \int_0^2 \left[16x - \frac{1}{3}x^3 - 2y^2x\right]_{x=0}^{x=2} dy \\
&= \int_0^2 \left(\frac{88}{3} - 4y^2\right) dy = \left[\frac{88}{3}y - \frac{4}{3}y^3\right]_0^2 = 48
\end{aligned}$$

Example 4. If $R = [0, \pi/2] \times [0, \pi/2]$, then, by Equation (4),
Solution.

$$\begin{aligned}
\int\int_R \sin x \cos y\, dA &= \int_0^{\pi/2} \sin x\, dx \int_0^{\pi/2} \cos y\, dy \\
&= [-\cos x]_0^{\pi/2} [\sin y]_0^{\pi/2} = 1 \cdot 1 = 1
\end{aligned}$$