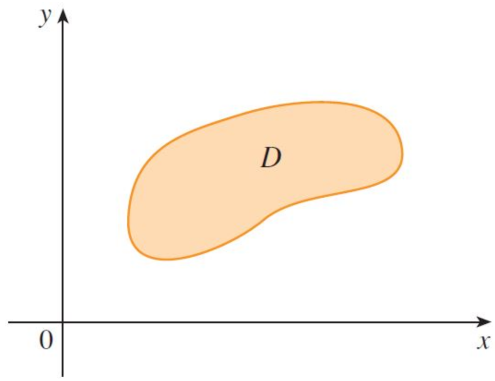
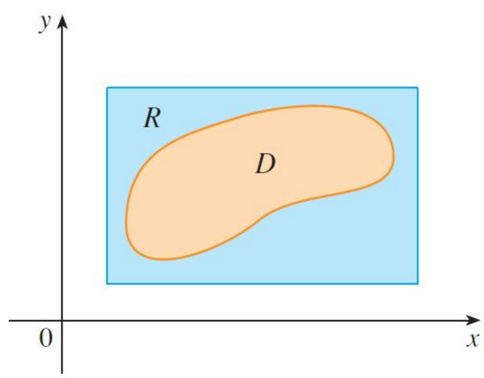
Double Integrals over General Region

In the last class, we learned about iterated integrals and the domain given for x and y is in the form of intervals. So, at most of the time, we can construct the domain, let say R, in the shape of a rectangle. So, what if R is just a general region that can take any shape like circle, or even a random shape as in figure below.



We suppose that D is a bounded region, which means that D can be enclosed in a rectangular region R as in figure below. Then we define a new function F with domain R by

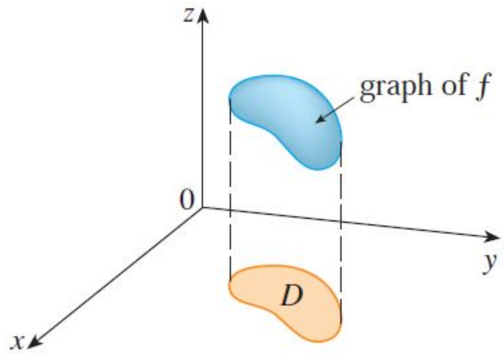
$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ is in } D \\ 0 & \text{if } (x,y) \text{ is in } R \text{ but not in } D \end{cases} \tag{1}$$



If F is integrable over R, then we define the **double integral of f over D** by

$$\iint\limits_D f(x,y)\,dA = \iint\limits_R F(x,y)\,dA \tag{2}$$

In the case where $f(x,y) \geq 0$, we can still interpret $\iint\limits_D f(x,y)\,dA$ as the volume of the solid that lies above D and under the surface $z = f(x,y)$ (the graph of f).

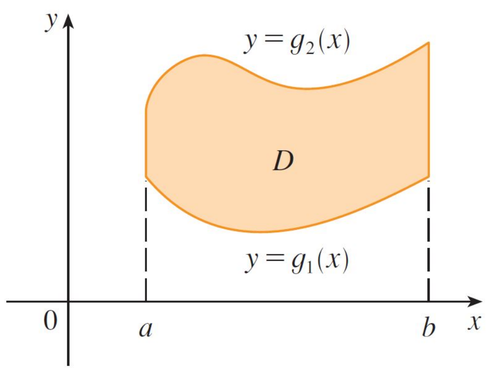


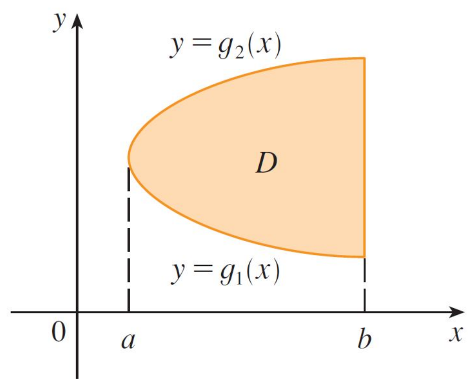
If f is continuous on D, then it can be shown that $\iint_R F(x,y)\,dA$ exists and therefore $\iint_D f(x,y)\,dA$ exists (with some exceptions).

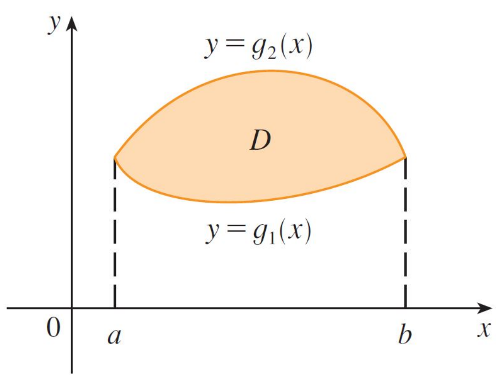
A plane region D is said to be of **type I** if it lies between the graphs of two continuous functions of x, that is,

$$D = \{(x,y)|a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

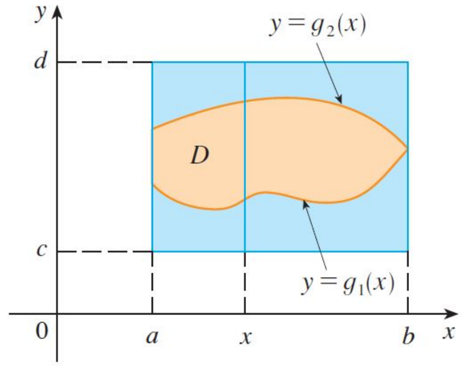
where g_1 and g_2 are continuous on $[a,b]$. Some example of this type can be refer in figures below.







In order to evaluate $\iint_D f(x,y)\,dA$ when D is a region of type I, we choose a rectangle $R = [a,b] \times [c,d]$ that contains D, as in figure below, and we let F be the function given by Equation (1); that is, F agrees with f on D and is 0 outside D.



Then, by Fubini's Theorem,

$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA = \int_a^b \int_c^d F(x,y)\,dy\,dx$$

Observe that $F(x,y) = 0$ if $y \le g_1(x)$ or $y \ge g_2(x)$ because (x,y) then lies outside D. Therefore

$$\int_c^d F(x,y)\,dy = \int_{g_1(x)}^{g_2(x)} F(x,y)\,dy = \int_{g_1(x)}^{g_2(x)} \quad dy$$

because $F(x,y) = f(x,y)$ when $g_1(x) \le y \le g_2(x)$. Thus we have the following formula that enables us to evaluate the double integral as an iterated integral.

If f is continuous on type I region D such that

$$D = \{(x,y)|a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

Then

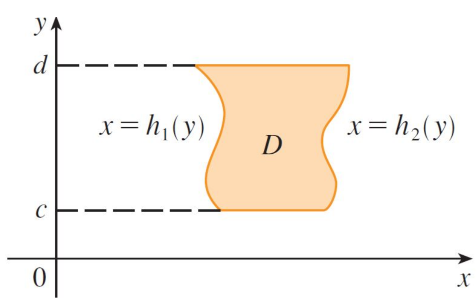
$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx \tag{3}$$

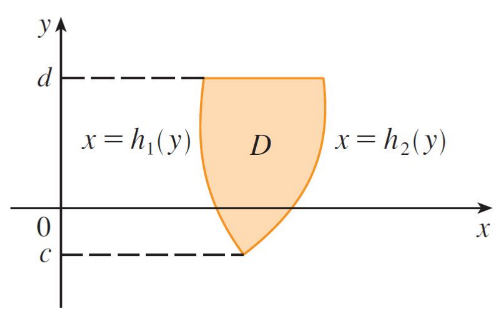
In the inner integral we regard x as being constant not only in $f(x,y)$ but also in the limits of integration, $g_1(x)$ and $g_2(x)$.

Next, we consider plane regions of **type II**, which can be expressed as

$$D = \{(x,y)|c \le y \le d, h_1(y) \le x \le h_2(y)\} \tag{4}$$

where $h_1(y)$ and $h_2(y)$ are continuous. It can be geometrically interpret as in the following figures.





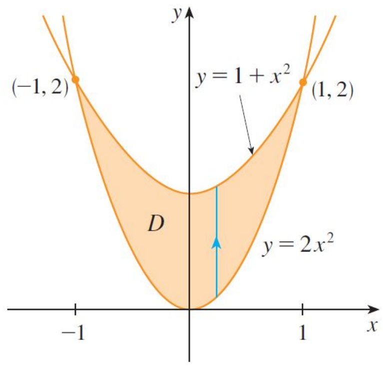
Similar to type I, we have

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy \tag{5}$$

where D is a type II region given by Equation (4).

Example 1. Evaluate $\iint\limits_D (x+2y)\,dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

Solution. The parabolas intersect when $2x^2 = 1 + x^2$, that is,$x^2 = 1$, so $x = \pm 1$. We note that the region D, sketched in figure below, is a type I region but not a type II region.



We can write

$$D = \{(x,y) | -1 \leqslant x \leqslant 1, 2x^2 \leqslant y \leqslant 1 + x^2\}$$

Since the lower boundary is $y = 2x^2$ and the upper boundary is $y = 1 + x^2$, Equation (3) gives

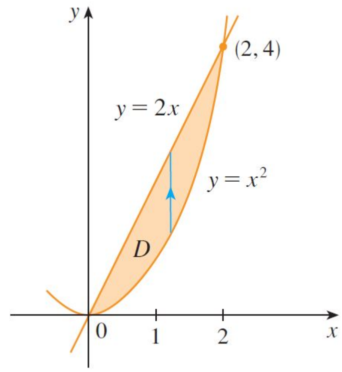
$$\begin{aligned}
\iint\limits_D (x+2y)\,dA &= \int_{-1}^{1}\int_{2x^2}^{1+x^2} (x+2y)\,dy\,dx \\
&= \int_{-1}^{1} \left[xy + y^2\right]_{y=2x^2}^{y=1+x^2}\,dx \\
&= \int_{-1}^{1} [x(1+x^2) + (1+x^2)^2 - x(2x^2) - (2x^2)^2]\,dx \\
&= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)\,dx \\
&= \left[-3\frac{x^5}{5} - \frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} + x\right]_{-1}^{1} = \frac{32}{15}
\end{aligned}$$

Note 1. *We can see that it is essential to draw a diagram. Often it is helpful to draw a vertical arrow as in figure in the previous example. Then the limits of integration for the inner integral can be read from the diagram as follows: The arrow starts at the lower boundary* $y = g_1(x)$*, which gives the lower limit in the integral, and the arrow ends at the upper boundary* $y = g_2(x)$*, which gives the upper limit of integration. For a type II region the arrow is drawn horizontally from the left boundary to the right boundary.*

Example 2. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

Solution. We shall provide two solutions for this question. The first one consider D as type I region, while the second one as type II region.

(1) First, we will consider D to be type I region.



We can write D as

$$D = \{(x, y) | 0 \leqslant x \leqslant 2, x^2 \leqslant y \leqslant 2x\}$$

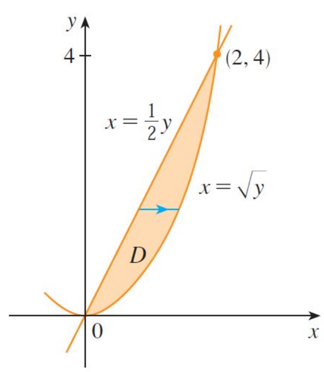
Therefore the volume under $z = x^2 + y^2$ and above D is

$$V = \iint_D (x^2 + y^2)\, dA = \int_0^2 \int_{x^2}^{2x} (x^2 + y^2)\, dy\, dx$$

$$= \int_0^2 \left[x^2 y + \frac{y^3}{3} \right]_{y=x^2}^{y=2x} dx = \int_0^2 \left[x^2(2x) + \frac{(2x)^3}{3} - x^2 x^2 - \frac{(x^2)^3}{3} \right] dx$$

$$= \int_0^2 \left(-\frac{x^6}{3} - x^4 + \frac{14x^3}{3} \right) dx = \left[\frac{x^7}{21} - \frac{x^5}{5} + \frac{7x^4}{6} \right]_0^2 = \frac{216}{35}$$

(2) For this solution, we consider D to be type II region.



We see that D can also be written as:

$$D = \{(x, y) | 0 \leqslant y \leqslant 4, \frac{1}{2}y \leqslant x \leqslant \sqrt{y}\}$$

Therefore another expression for V is

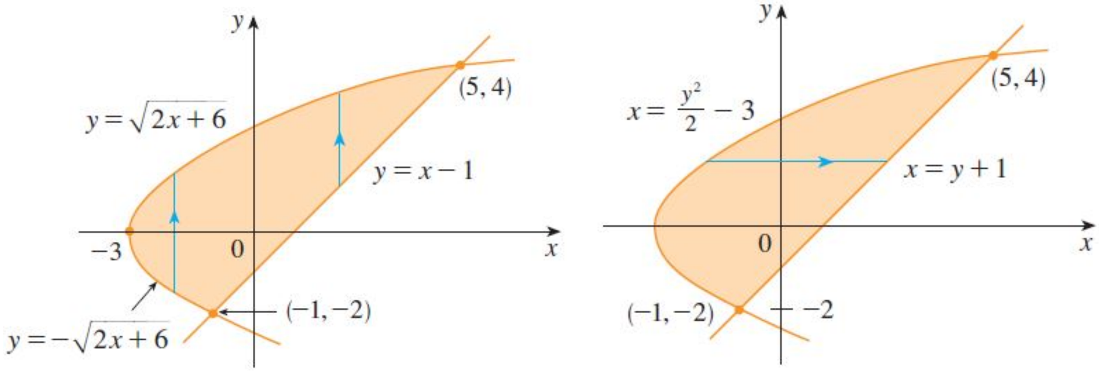
$$V = \iint_D (x^2 + y^2)\, dA = \int_0^4 \int_{(1/2)y}^{\sqrt{y}} (x^2 + y^2)\, dx\, dy$$

$$= \int_0^4 \left[\frac{x^3}{3} + y^2 x \right]_{x=(1/2)y}^{x=\sqrt{y}} dy = \int_0^4 \left(\frac{y^{3/2}}{3} + y^{5/2} - \frac{y^3}{24} - \frac{y^3}{2} \right) dy$$

$$= \left[\frac{2}{15} y^{5/2} + \frac{2}{7} y^{7/2} - \frac{13}{96} y^4 \right]_0^4 = \frac{216}{35}$$

Example 3. Evaluate $\iint\limits_D xy\,dA$ where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

Solution. The region D is shown in figure below.



Again D is both type I and type II, but the description of D as a type I region is more complicated because the lower boundary consists of two parts. Therefore we prefer to express D as a type II region:

$$D = \{(x, y)| - 2 \leqslant y \leqslant 4, \frac{1}{2}y^2 - 3 \leqslant x \leqslant y + 1\}$$

Then Equation (5) gives

$$\begin{aligned}
\iint\limits_D xy\,dA &= \int_{-2}^{4}\int_{(1/2)y^2+3}^{y+1} xy\,dx\,dy = \int_{-2}^{4}\left[\frac{x^2}{2}y\right]_{x=(1/2)y^2+3}^{x=y+1} dy \\
&= \frac{1}{2}\int_{-2}^{4} y\left[(y+1)^2 - (\frac{1}{2}y^2 - 3)^2\right] dy \\
&= \frac{1}{2}\int_{-2}^{4}\left(-\frac{y^5}{4} + 4y^3 + 2y^2 - 8y\right) dy \\
&= \frac{1}{2}\left[-\frac{y^6}{24} + y^4 + 2\frac{y^3}{3} - 4y^2\right]_{-2}^{4} = 36
\end{aligned}$$

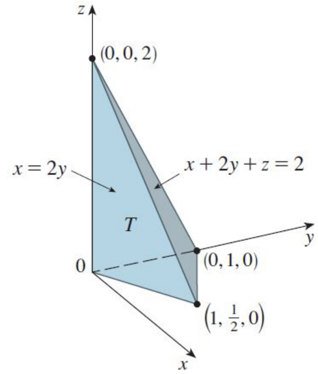
If we had expressed D as a type I region using figure on the left above, then we would have obtained

$$\iint\limits_D xy\,dA = \int_{-3}^{-1}\int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} xy\,dy\,dx + \int_{-1}^{5}\int_{x-1}^{\sqrt{2x+6}} xy\,dy\,dx$$

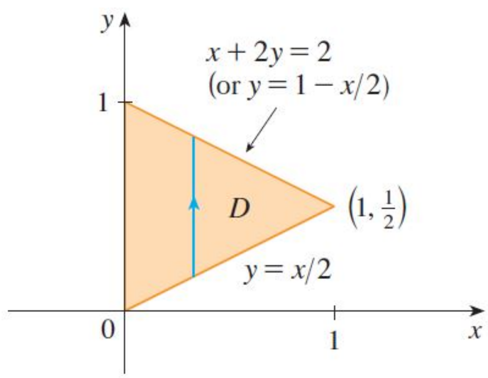
but this would have involved more work than the other method.

Example 4. Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

Solution. In a question such as this, its wise to draw two diagrams: one of the three dimensional solid and another of the plane region D over which it lies. Figure below shows the tetrahedron T bounded by the coordinate planes $x = 0$, $z = 0$, the vertical plane $x = 2y$, and the plane $x + 2y + z = 2$.



Since the plane $x + 2y + z = 2$ intersects the xy-plane (whose equation is $z = 0$) in the line $x + 2y = 2$, we see that T lies above the triangular region D in the xy-plane bounded by the lines $x = 2y$, $x + 2y = 2$, and $x = 0$. This can be geometrically represented as in the following figure.



The plane $x + 2y + z = 2$ can be written as $z = 2 - x - 2y$, so the required volume lies under the graph of the function $z = 2 - x - 2y$ and above

$$D = \{(x, y)|0 \leqslant x \leqslant 1, \frac{x}{2} \leqslant y \leqslant 1 - \frac{x}{2}\}$$

Therefore

$$\begin{aligned} V &= \iint_D (2 - x - 2y)\, dA = \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y)\, dy\, dx \\ &= \int_0^1 \left[2y - xy - y^2\right]_{y=x/2}^{y=1-x/2} dx \\ &= \int_0^1 \left[2 - x - x\left(1 - \frac{x}{2}\right) - \left(1 - \frac{x}{2}\right)^2 - x + \frac{x^2}{2} + \frac{x^2}{4}\right] dx \\ &= \int_0^1 (x^2 - 2x + 1)\, dx = \left[\frac{x^3}{3} - x^2 + x\right]_0^1 = \frac{1}{3} \end{aligned}$$

Example 5. Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$.

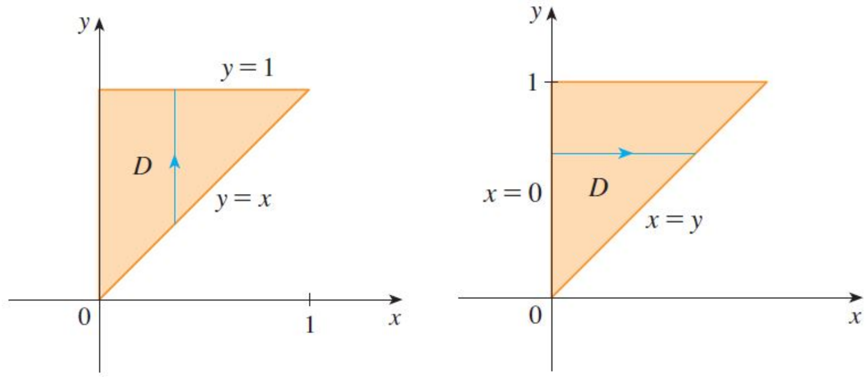
Solution. If we try to evaluate the integral as it stands, we are faced with the task of first evaluating $\int \sin(y^2)\, dy$. But its impossible to do so in finite terms since $\int \sin(y^2)\, dy$ is not an elementary function. So we must change the order of integration. This is accomplished by first expressing the given iterated integral as a double integral. Using Equation (3) backward, we have

$$\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx = \iint_D \sin(y^2)\,dA$$

where

$$D = \{(x,y)|0 \leqslant x \leqslant 1, x \leqslant y \leqslant 1\}$$

We can sketch this region D in the figure on the left below.



Then from figure on the right, we see that an alternative description of D is

$$D = \{(x,y)|0 \leqslant y \leqslant 1, 0 \leqslant x \leqslant y\}$$

This enables us to use Equation (5) to express the double integral as an iterated integral in the reverse order:

$$\begin{aligned}\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx &= \iint_D \sin(y^2)\,dA \\ &= \int_0^1 \int_0^y \sin(y^2)\,dx\,dy = \int_0^1 \left[x\sin(y^2)\right]_{x=0}^{x=y} dy \\ &= \int_0^1 y\sin(y^2)\,dy = -\left[\frac{1}{2}\cos(y^2)\right]_0^1 \\ &= \frac{1}{2}(1-\cos 1)\end{aligned}$$

Properties of Double Integrals

We assume that all of the following integrals exist. The first three properties of double integrals over a region D follow immediately from the knowledge on multiple integrals.

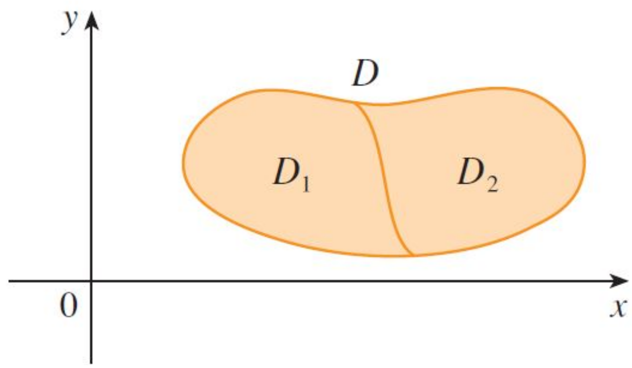
$$\iint_D [f(x,y)+g(x,y)]\,dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA \tag{6}$$

$$\iint_D cf(x,y)\,dA = \iint_D f(x,y)\,dA \tag{7}$$

If $f(x,y) \geq g(x,y)$ for all (x,y) in D, then

$$\iint_D f(x,y)\,dA \geq \iint_D g(x,y)\,dA \tag{8}$$

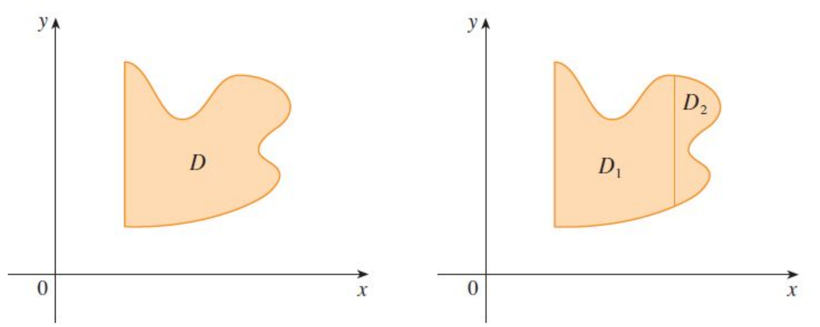
The next property of double integrals is similar to the property of single integrals given by the equation $\int_a^b f(x)\,dx = \int_a^c f(x,y)\,dA + \int_c^b f(x,y)\,dx$. Consider $D = D_1 \bigcup D_2$, where D_1 and D_2 dont overlap except perhaps on their boundaries as in figure below.



Then

$$\boxed{\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA} \tag{9}$$

Property 9 can be used to evaluate double integrals over regions that are neither type I nor type II but can be expressed as a union of regions of type I or type II. Figure below illustrates this procedure.

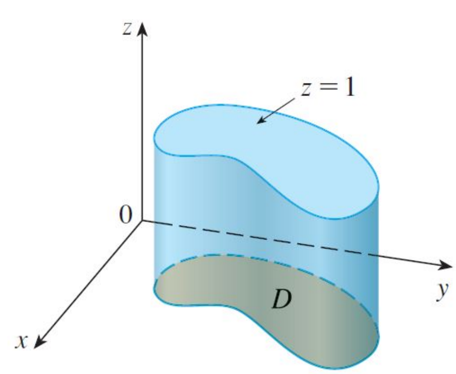


(a) D is neither type I nor type II. (b) $D = D_1 \cup D_2$, D_1 is type I, D_2 is type II.

The next property of integrals says that if we integrate the constant function $f(x,y) = 1$ over a region D, we get the area of D:

$$\boxed{\iint_D 1\,dA = A(D)} \tag{10}$$

Figure below illustrates why Equation (10) is true: A solid cylinder whose base is D and whose height is 1 has volume $A(D).1 = A(D)$, but we know that we can also write its volume as $\iint_D 1\,dA$.



Finally, combining Equations (6), (8), (10), we can prove the following property:

If $n \leq f(x, y) \leq M$ *for all* (x, y) *in* D*, then*

$$mA(D) \leq \iint_D f(x, y)\, dA \leq MA(D) \tag{11}$$

Example 6. Use Property 11 to estimate the integral $\iint_D e^{\sin x \cos y}\, dA$, where D is the disk with center the origin and radius 2.

Solution. Since $-1 \leqslant \sin x \leqslant 1$ and $-1 \leqslant \cos y \leqslant 1$, we have $-1 \leqslant \sin x \cos y \leqslant 1$ and therefore

$$e^{-1} \leqslant e^{\sin x \cos y} \leqslant e^1 = e$$

Thus, using $m = e^{-1} = 1/e$, $M = e$, and $A(D)\pi(2)^2$ in Property 11, we obtain

$$\frac{4\pi}{e} \leqslant \iint_D e^{\sin x \cos y}\, dA \leqslant 4\pi e$$