Double Integrals in Polar Coordinates

Suppose that we want to evaluate a double integral $\iint_R f(x,y)\,dA$, where R is one of the regions shown in Figure 1. In either case the description of R in terms of rectangular coordinates is rather complicated but R is easily described using polar coordinates.

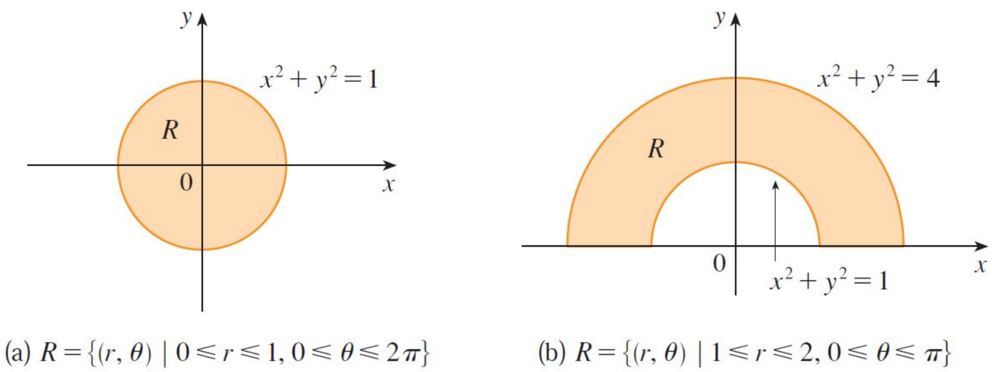


Figure 1. Region that best described using polar coordinates

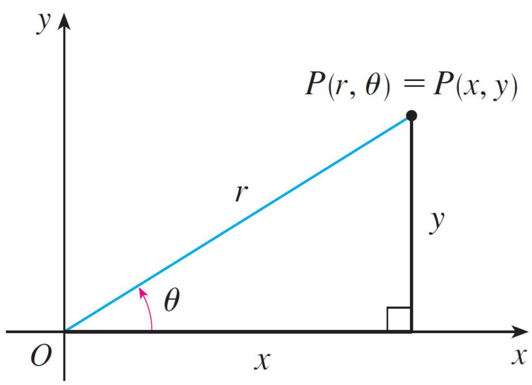


Figure 2. Relation between x and y with r and θ

Recall from Figure 2 that the polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) by the equations

$$\boxed{r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta}$$

The regions in Figure 1 are special cases of a **polar rectangle**

$$R = \{(r, \theta) | a \leq r \leq b, \alpha \leq \theta \leq \beta\}$$

as shown in Figure 3. In order to compute the double integral $\iint_R f(x,y)\,dA$, where R is a polar rectangle, we divide the interval $[a, b]$ into m subintervals of equal width $\Delta r = (b - a)/m$ and we divide the interval $[\alpha, \beta]$ into n subintervals $[\theta_{j-1}, \theta_j]$ of equal width $\Delta\theta = (\beta - \alpha)/n$. Then the circles $r = r_i$ and the rays $\theta = \theta_j$ divide the polar rectangle R into the small polar rectangles shown in Figure 4.
The "center" (from the idea of midpoint) of the polar subrectangle

$$R_{ij} = \{(r, \theta) | r_{i-1} \leq r \leq r_i, \theta_{j-1} \leq \theta\theta_j\}$$

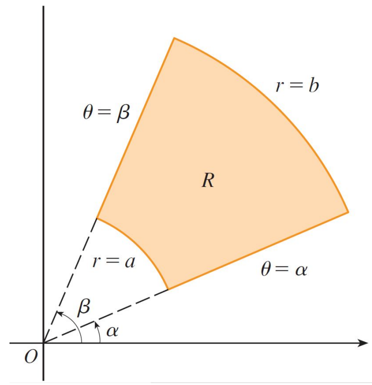


FIGURE 3. Polar rectangle

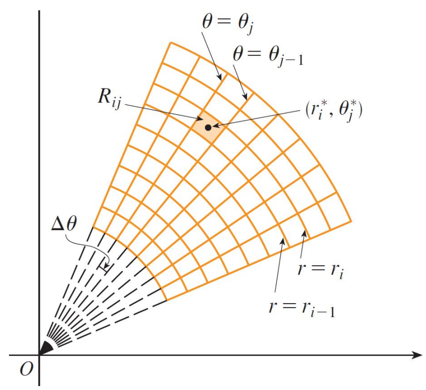


FIGURE 4. Polar subrectangles

has polar coordinates

$$r_i^* = \frac{1}{2}(r_{i-1} + r_i) \qquad \theta_j^* = \frac{1}{2}(\theta_{j-1} - \theta_j)$$

We compute the area of R_{ij} using the fact that the area of a sector of a circle with radius r and central angle θ is $\frac{1}{2}r^2\theta$. Subtracting the areas of two such sectors, each of which has central angle $\Delta\theta = \theta_j - \theta_{j-1}$, we find that the area of R_{ij} is

$$\begin{aligned}\Delta A_i &= \frac{1}{2}r_i^2\Delta\theta - \frac{1}{2}r_{i-1}^2\Delta\theta = \frac{1}{2}(r_i^2 - r_{i-1}^2)\Delta\theta \\ &= \frac{1}{2}(r_i - r_{i-1})(r_i + r_{i-1})\Delta\theta = r_i^*\Delta r\Delta\theta\end{aligned}$$

Although we have defined the double integral $\iint_R f(x,y)\,dA$ in terms of ordinary rectangles, it can be shown that, for continuous functions f, we always obtain the same answer using polar rectangles. The rectangular coordinates of the center of R_{ij} are $(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)$, so a typical Riemann sum is

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)\Delta A_i = \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)r_i^*\Delta r\Delta\theta \tag{1}$$

If we write $g(r,\theta) = rf(r\cos\theta, r\sin\theta)$, then the Riemann sum in Equation (1) can be written as

$$\sum_{i=1}^{m}\sum_{j=1}^{n} g(r_i^*, theta_j^*)\Delta r\Delta\theta$$

which is a Riemann sum for the double integral

$$\int_\alpha^\beta \int_a^b g(r,\theta)\,dr\,d\theta$$

Therefore,

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\Delta A_i$$
$$= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} g(r_i^*, theta_j^*)\Delta r \Delta\theta = \int_\alpha^\beta \int_a^b g(r,\theta)\,dr\,d\theta$$
$$= \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$$

So, we have the following theorem to **change to polar coordinates in a double integral**

Theorem 1. *If f is continuous on a polar rectangle R given by $0 \le a \le r \le b, \alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then*

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

The formula in Theorem 1 says that we convert from rectangular to polar coordinates in a double integral by writing $x = r\cos\theta$ and $y = r\sin\theta$, using the appropriate limits of integration for r and θ, and replacing dA by $r\,dr\,d\theta$. Be careful not to forget the additional factor r on the right side of Formula in Theorem 1. A classical method for remembering this is shown in Figure 5, where the infinitesimal polar rectangle can be thought of as an ordinary rectangle with dimensions $r\,d\theta$ and dr and therefore has area $dA = r\,dr\,d\theta$.

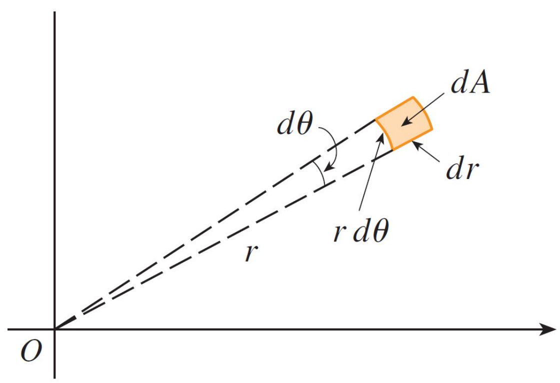


FIGURE 5. Infinitesimal polar rectangle

Example 1. Evaluate $\int\int_R (3x + 4y^2)\,dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution. The region R can be described as

$$R = \{(x,y) | y \geqslant 0, 1 \leqslant x^2 + y^2 \leqslant 4\}$$

It is the half-ring shown in Figure 2, and in polar coordinates it is given by $1 \leqslant r \leqslant 2$, $0 \leqslant \theta \leqslant \pi$. Therefore, by formula in Theorem 1,

$$\begin{aligned}
\int\int_R (3x + 4y^2)\, dA &= \int_0^\pi \int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\, r\, dr\, d\theta \\
&= \int_0^\pi \int_1^2 (3r^2\cos\theta + 4r^3\sin^2\theta)\, dr\, d\theta \\
&= \int_0^\pi \left[r^3\cos\theta + r^4\sin^2\theta\right]_{r=1}^{r=2} d\theta = \int_0^\pi (7\cos\theta + 15\sin^2\theta)\, d\theta \\
&= \int_0^\pi \left[7\cos\theta + \frac{15}{2}(1 - \cos 2\theta)\right] d\theta \\
&= \left[7\sin\theta + \frac{15\theta}{2} - \frac{15}{4}\sin 2\theta\right]_0^\pi = \frac{15\pi}{2}
\end{aligned}$$

Example 2. Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

Solution. If we put $z = 0$ in the equation of the paraboloid, we get $x^2 + y^2 = 1$. This means that the plane intersects the paraboloid in the circle $x^2 + y^2 = 1$, so the solid lies under the paraboloid and above the circular disk D given by $x^2 + y^2 \leqslant 1$. In polar coordinates D is given by $0 \leqslant r \leqslant 1$, $0 \leqslant \theta \leqslant 2\pi$. Since $1 - x^2 - y^2 = 1 - r^2$, the volume is

$$\begin{aligned}
V &= \int\int_D (1 - x^2 - y^2)\, dA = \int_0^{2\pi} \int_0^1 (1 - r^2)\, r\, dr\, d\theta \\
&= \int_0^{2\pi} d\theta \int_0^1 (r - r^3)\, dr = 2\pi \left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 = \frac{\pi}{2}
\end{aligned}$$

If we had used rectangular coordinates instead of polar coordinates, then we would have obtained

$$V = \int\int_D (1 - x^2 - y^2)\, dA = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)\, dy\, dx$$

which is not easy to evaluate because it involves finding $\int (1 - x^2)^{3/2}\, dx$.

What we have done so far can be extended to the more complicated type of region shown in Figure 6.

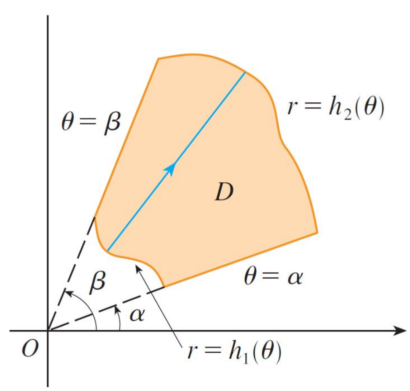


FIGURE 6. Region enclosed

Its similar to the type II rectangular regions considered in double integral over general region. In fact, by combining formula in Theorem 1 in this section with for type II rectangular regions, we obtain the following formula.

Theorem 2. *If f is continuous on a polar region of the form*

$$D = \{(r,\theta)|\alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$$

then,

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

In particular, taking $f(x,y) = 1$, $h_1(\theta) = 0$, and $h_2(\theta) = h(\theta)$ in this formula, we see that the area of the region D bounded by $\theta = \alpha$, $\theta = \beta$, and $r = h(\theta)$ is

$$A(D) = \iint_D 1\,dA = \int_\alpha^\beta \int_0^{h(\theta)} r\,dr\,d\theta$$
$$= \int_\alpha^\beta \left[\frac{r^2}{0}\right]^h (\theta)_0\,d\theta = \int_\alpha^\beta \frac{1}{2}[h(\theta)]^2\,d\theta$$

Example 3. Use a double integral to find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

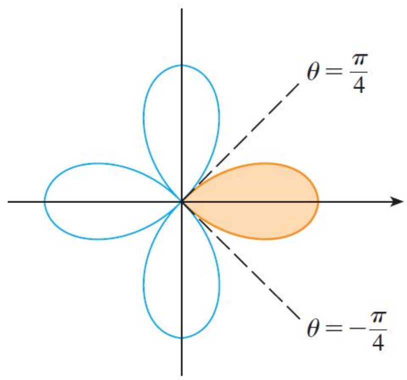


FIGURE 7. Sketch of the enclosed area

Solution. From the sketch of the curve in Figure 7, we see that a loop is given by the region

$$D = \{(r,\theta)| -\pi/4 \leqslant \theta \leqslant \pi/4, 0 \leqslant r \leqslant \cos 2\theta\}$$

So the area is

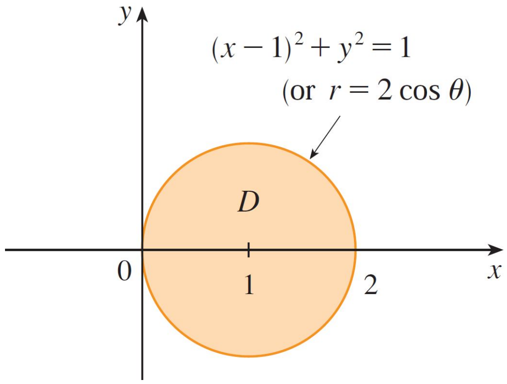
$$A(D) = \int\int_D dA = \int_{-\pi/4}^{\pi/4}\int_0^{\cos 2\theta} r\,dr\,d\theta$$
$$= \int_{-\pi/4}^{\pi/4}\left[\frac{1}{2}r^2\right]_0^{\cos 2\theta} d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4}\cos^2 2\theta\,d\theta$$
$$= \frac{1}{4}\int_{-\pi/4}^{\pi/4}(1+\cos 4\theta)\,d\theta = \frac{1}{4}\left[\theta + \frac{1}{4}\sin 4\theta\right]_{-\pi/4}^{\pi/4} = \frac{\pi}{8}$$

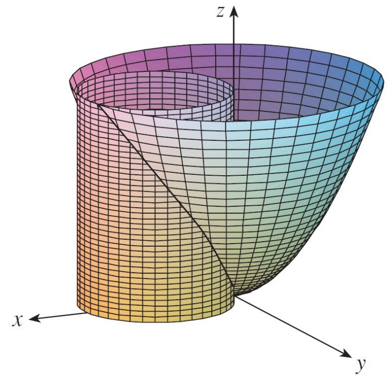
Example 4. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$,above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

Solution. The solid lies above the disk D whose boundary circle has equation $x^2 + y^2 = 2x$ or, after completing the square,

$$(x-1)^2 + y^2 = 1$$

In polar coordinates we have $x^2 + y^2 = r^2$ and $x = r\cos\theta$, so the boundary circle becomes $r^2 = 2r\cos\theta$, or $r = 2\cos\theta$ (See figures below).





Thus the disk D is given by

$$D = \{(r,\theta) | -\pi/2 \leqslant \theta \leqslant \pi/2, 0 \leqslant r \leqslant 2\cos\theta\}$$

and, by Formula 3, we have

$$\begin{aligned} V &= \int\int_D (x^2+y^2)\,dA = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^2 r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\left[\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta \\ &= 4\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta = 8\int_0^{\pi/2}\cos^4\theta\,d\theta = 8\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta \\ &= 2\int_0^{\pi/2}\left[1+2\cos 2\theta+\frac{1}{2}(1+\cos 4\theta)\right] d\theta \\ &= 2\left[\frac{3}{2}\theta+\sin 2\theta+\frac{1}{8}\sin 4\theta\right]_0^{\pi/2} = 2\left(\frac{3}{2}\right)\left(\frac{\pi}{2}\right) = \frac{3\pi}{2} \end{aligned}$$