Triple Integrals

We have defined single integrals for functions of one variable and double integrals for functions of two variables, so we can define triple integrals for functions of three variables.
Let's first deal with the simplest case where f is defined on a rectangular box:

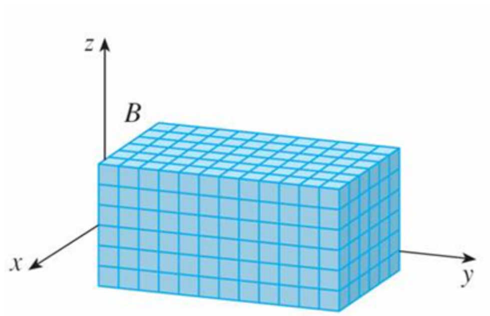
$$B = \{(x, y, z)|a \leq x \leq b, r \leq z \leq s\} \tag{1}$$

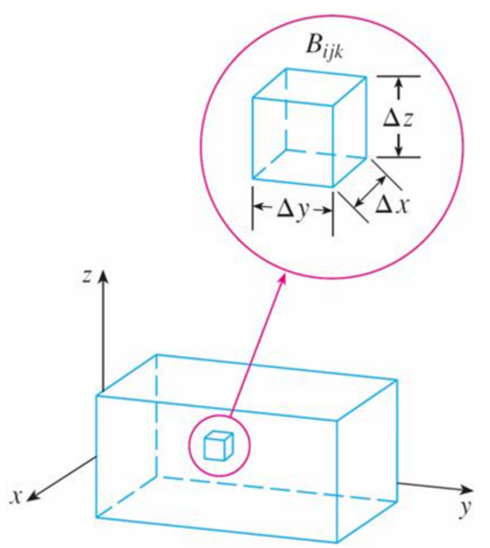
The first step is to divide B into sub-boxes. We do this by dividing the interval $[a, b]$ into l subintervals $[x_{i-1}, x_i]$ of equal width Δx, dividing $[c, d]$ into m subintervals of width Δy, and dividing $[r, s]$ into n subintervals of width Δz.

The planes through the endpoint of these subintervals parallel to the coordinate planes divide the box B into lmn sub-boxes

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$

as shown in the following figures:





Each sub-box has volume $\Delta V = \Delta x \Delta y \Delta z$. Then we form the **triple Riemann sum**

$$\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \Delta V \tag{2}$$

where the sample point $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ is in B_{ijk}.

By analogy with the definition of a double integral, we define the triple integral as the limit of the triple Riemann sums in 2.

Definition 1. The **triple integral** of f over the box B is

$$\int\int\int_B f(x,y,z)\,dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\Delta V$$

Again, the triple integral always exists if f is continuous. We can choose the sample point to be any point in the sub-box, but if we choose it to be the point (x_i, y_j, z_k), we get a simpler looking expression for the triple integral

$$\int\int\int_B f(x,y,z)\,dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_i, y_j, z_k)\Delta V$$

Just as for double integrals, the practical method for evaluating triple integrals is to express them as iterated integrals as follows.

Theorem 1. Fubini's Theorem for Triple Integrals *If f is continuous on the rectangular box $B = [a,b]\times[c,d]\times[r,s]$, then*

$$\int\int\int_B f(x,y,z)\,dV = \int_r^s\int_c^d\int_a^b f(x,y,z)\,dx\,dy\,dz$$

The iterated integral on the right side of Fubini's Theorem means that we integrate first with respect to x (keeping y and z fixed), then z fixed), and finally we integrate with respect to z.

There are five other possible orders in which we can integrate, all of which give the same value. For instance, if we integrate with respect to y, then z, and then x, we have

$$\int\int\int_B f(x,y,z)\,dV = \int_a^b\int_r^s\int_c^d f(x,y,z)\,dy\,dz\,dx$$

Example 1. Evaluate the triple integral $\int\int\int_B xyz^2\,dV$ where B is the rectangular box given by

$$B = (x,y,z)|0\le x\le 1, -1\le y\le 2, 0\le z\le 3$$

Solution. We could use any of the six possible orders of integration.

If we choose to integrate with respect to x, then y, and then z, we obtain

$$\int\int\int_B xyz^2\, dV = \int_0^3 \int_{-} 1^2 \int_0^1 xyz^2\, dx\, dy\, dz$$
$$= \int_0^3 \int_{-1}^2 \left[\frac{x^2yz^2}{2}\right]_{x=0}^{x=1} dy\, dz$$
$$= \int_0^3 \int_{-1}^2 \frac{yz^2}{2}\, dy\, dz$$
$$= \int_0^3 \left[\frac{y^2z^2}{4}\right]_{y=-1}^{y=2} dz$$
$$= \int_0^3 \frac{3z^2}{4}\, dz$$
$$= \left.\frac{z^3}{4}\right] = \frac{27}{4}$$

Now we define the **triple integral over a general bounded region** E in three-dimensional space (a solid) by much the same procedure that we used for double integrals.

We enclose E in a box B of the type given by Equation 1. Then we define F so that it agrees with f on E but is 0 for points in B that are outside E.

By definition,

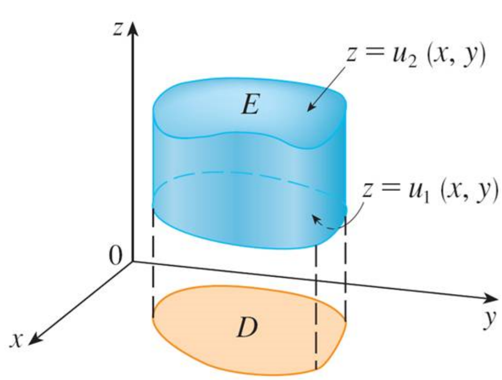
$$\int\int\int_E f(x, y, z)\, dV = \int\int\int_B F(x, y, z)\, dV$$

This integral exist if f is continuous and the boundary of E is "reasonably smooth". The triple integral has essentially the same properties as the double integral. We restrict our attention to continuous functions f and to certain simple types of regions.

A solid region E is said to be of **type 1** if it lies between the graphs of two continuous functions of x and y, that is,

$$E = \{(x, y, z)|(x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\} \tag{3}$$

where D is the projection of E onto the xy-plane as shown in the following figure.



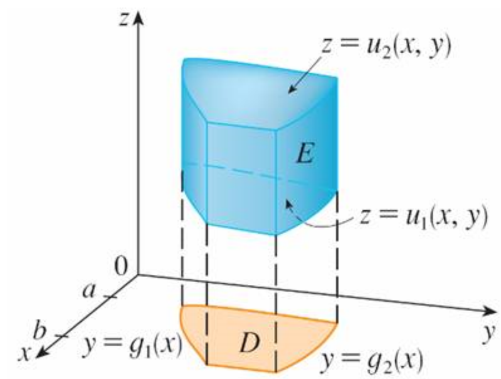
Notice that the upper boundary of the solid E is the surface with equation $z = u_2(x, y)$, while the lower boundary is the surface $z = u_1(x, y)$.

By the same sort of argument, it can be shown that if E is a type 1 region given by Equation 3, then

$$\int\int\int_E f(x, y, z)\, dV = \int\int_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz \right] dA \tag{4}$$

The meaning of the inner integral on the right side of Equation 4 is that x and y are held fixed, and therefore $u_1(x, y)$ and $u_2(x, y)$ are regarded as constants, while $f(x, y, z)$ is integrated with respect to z.

In particular, if the projection D of E onto the xy-plane is a type I plane region (as in figure below).

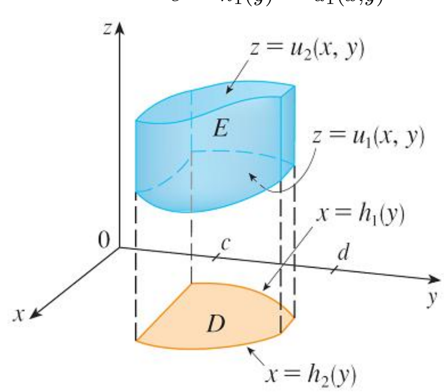


Then $E = \{(x, y, z)| a \leq x \leq b, g_1(x) \leq y \leq g_2(x), u_1(x, y) \leq z \leq u_2(x, y)\}$ and Equation 4 becomes

$$\int\int\int_E f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\, dy\, dx \tag{5}$$

If, on the other hand, D is a type II plane region (as in figure below), then $E = \{(x, y, z)| c \leq y \leq d, h_1(y) \leq x \leq h_2(y), u_1(x, y) \leq z \leq u_2(x, y)\}$ and Equation 4 becomes

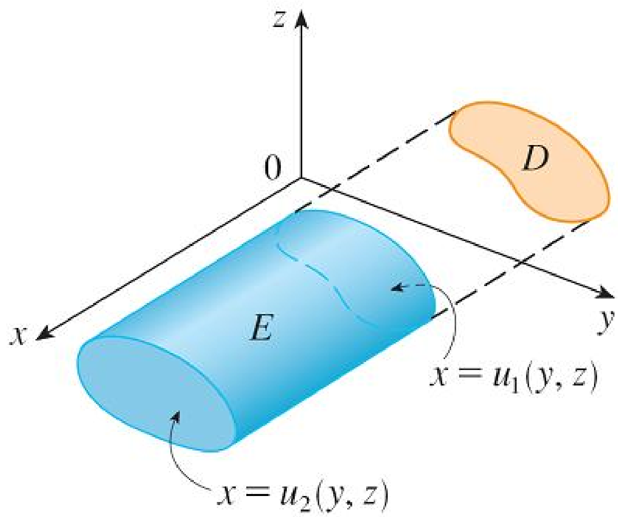
$$\int\int\int_E f(x, y, z)\, dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\, dx\, dy \tag{6}$$



A solid region E is of type 2 if it is of the form

$$E = \{(x, y, z)| (y, z) \in D, u_1(y, z) \leq x \leq u_2(y, z)\}$$

where, this time, D is the projection of E onto the yz-plane (See figure below).



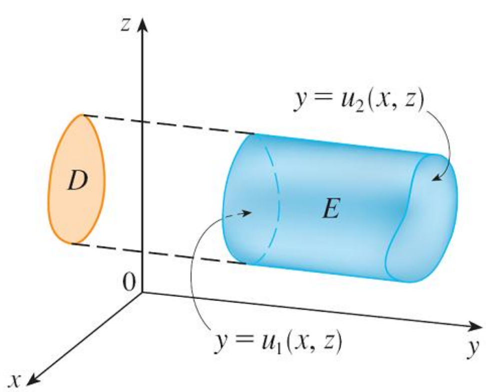
The back surface is $x = u_1(y,z)$, the front surface is $x = u_2(y,z)$, and we have

$$\int\int\int_E f(x,y,z)\,dV = \int\int_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\right] dA \tag{7}$$

Finally, a **type 3** region is of the form

$$E = \{(x,y,z)|(x,z) \in D, u_1(x,z) \leq y \leq u_2(x,z)\}$$

where D is the projection of E onto the xz-plane, $y = u_1(x,z)$ is the left surface, and $y = u_2(x,z)$ is the right surface (see figure below).



For this type of region we have

$$\int\int\int_E f(x,y,z)\,dV = \int\int_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\right] dA \tag{8}$$

In each of Equations 7 and 8 there may be two possible expressions for the integral depending on whether D is a type I or type II plane region (and corresponding to Equations 5 and 6).

Application of Triple Integrals

We know that if $f(x) \geq 0$, then the single integral $]int_a^b f(x)\,dx$ represents the area under the curve $y = f(x)$ from a to b, and of $f(x,y) \geq 0$, then the double integral $\int\int_D f(x,y)\,dA$ represents the volume under the surface $z = f(x,y)$ and above D.

The corresponding interpretation of a triple integral $\int\int\int_E f(x,y,z)\,dV$, where $f(x,y,z) \geq 0$, is not very useful because it would be hyper volume" of a four-dimensional object and, of course, that is very difficult to visualize.

Nonetheless, the triple integral $\int\int\int_E f(x,y,z)\,dV$ can be interpreted in different ways in different physical situations, depending on the physical interpretations of x, y, z and $f(x,y,z)$.

For example, a special case where $f(x,y,z) = 1$ for all points in E. Then, the triple integral does represent the volume of E:

$$V(E) = \int\int\int_E dV \tag{9}$$

You can see this in the case of type 1 region by putting $f(x,y,z) = 1$ in Formula 4:

$$\int\int\int_E 1\,dV = \int\int_D \left[\int_{u_1(x,y)}^{u_2 x, y\,dz}\right] dA = \int\int_D [u_2 x, y - u_1(x,y)]\,dA$$

and we know this represents the volume that lies between the surfaces $z = u_1(x,y)$ and $z = u_2(x,y)$.