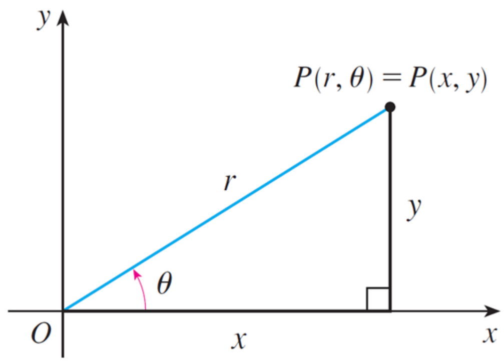
Triple Integrals in Cylindrical Coordinates

In plane geometry the polar coordinate system is used to give a convenient description of certain curves and regions.

Figure below enables us to recall the connection between polar and Cartesian coordinates.



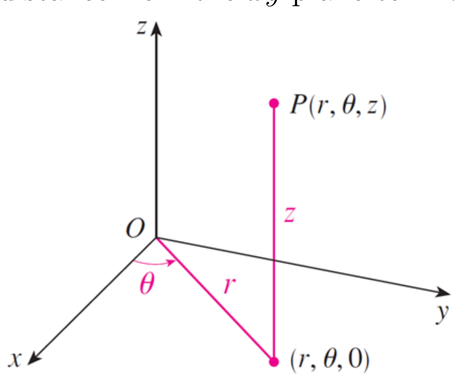
If the point P has Cartesian coordinates (x, y) and polar coordinates (r, θ), then, from the figure,

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ r^2 &= x^2 + y^2 \\ \tan\theta &= \frac{y}{x} \end{aligned}$$

In three dimensions there is a coordinate system, called cylindrical coordinates, that is similar to polar coordinates and gives convenient descriptions of some commonly occurring surfaces and solids. As we will see, some triple integrals are much easier to evaluate in cylindrical coordinates.

Cylindrical Coordinates

In the cylindrical coordinate system, a point P in three dimensional space is represented by the ordered triple (r, θ, z), where r and θ are polar coordinates of the projection of P onto the xy-plane and z is the directed distance from the xy-plane to P as in the following figure.



To convert from cylindrical to rectangular coordinates, we use the equations

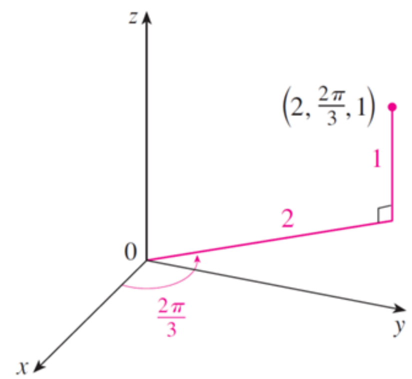
$$x = r\cos\theta,\ y = r\sin\theta,\ z = z \tag{1}$$

whereas to convert from rectangular to cylindrical coordinates, we use

$$r^2 = x^2 + y^2,\ \tan\theta = \frac{y}{x},\ z = z \tag{2}$$

Example 1. (1) Plot the point with cylindrical coordinates $(2, 2\pi/3, 1)$ and find its rectangular coordinates.
(2) Find cylindrical coordinates of the point with rectangular coordinates (3,–3,–7).

Solution. a) The point with the cylindrical coordinate $(2, 2\pi/3, 1)$ can be plotted as in the following figure.



From Equation 1, its rectangular coordinates are

$$\begin{aligned} x &= 2\cos\frac{2\pi}{3} = 2\left(-\frac{1}{2}\right) = -1 \\ x &= 2\sin\frac{2\pi}{3} = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3} \\ z &= 1 \end{aligned}$$

So, the point is $(-1, \sqrt{3}, 1)$ in rectangular coordinates.

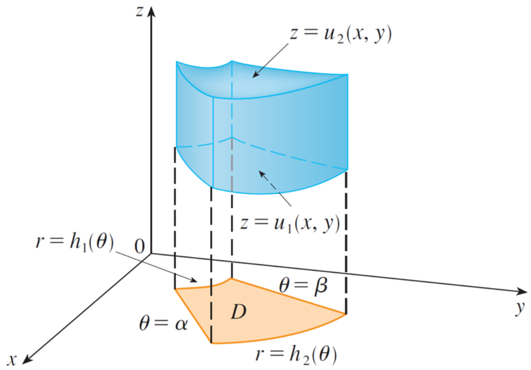
b) From Equation 2,

$$\begin{aligned} r &= \sqrt{3^2 + (-3)^2} = 3\sqrt{2} \\ \tan\theta = \frac{-3}{3} = -1 \implies \theta &= \frac{7\pi}{4} + 2n\pi \\ z &= -1 \end{aligned}$$

Therefore one set of cylindrical coordinates is $(3\sqrt{2}, 7\pi/4, -7)$.

Evaluating Triple Integrals in Cylindrical Coordinates

Suppose that E is a type 1 region whose projection D onto the xy-plane is conveniently described in polar coordinates as in the following figure.



In particular, suppose that f is continuous and

$$E = \{(x, y, z)|(x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$$

where D is given in polar coordinates by

$$D = \{(r, \theta)|\alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$$

We know

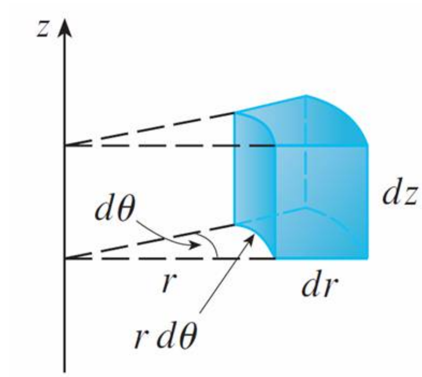
$$\int\int\int_E f(x, y, z)\, dV = \int\int_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\right] dA \tag{3}$$

But to evaluate double integrals in polar coordinates, we have the formula

$$\int\int\int_E f(x, y, z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta \tag{4}$$

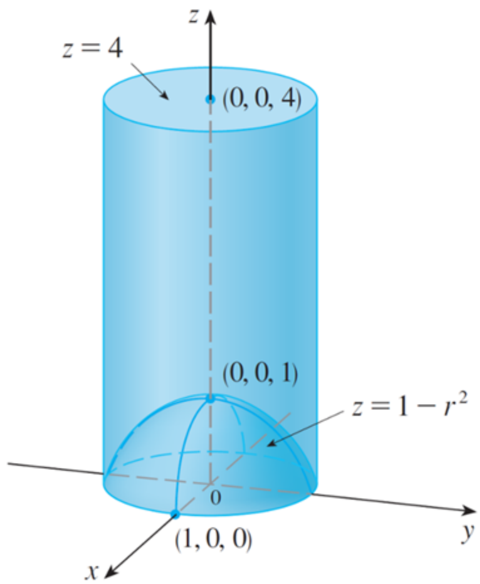
Formula 4 is the formula for triple integration in cylindrical coordinates.

It says that we convert a triple integral from rectangular to cylindrical coordinates by writing $x = r\cos\theta$, $y = r\sin\theta$, leaving z as it is, using the appropriate limits of integration for z, r, and θ and replacing dV by $r\, dz\, dr\, d\theta$.



It is worthwhile to use this formula when E is a solid region easily described in cylindrical coordinates, and especially when the function $f(x, y, z)$ involves the expression $x^2 + y^2$.

Example 2. A solid E lies within the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$. (See figure below.) The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E.



In cylindrical coordinates the cylinder is $r = 1$ and the paraboloid is $z = 1 - r^2$, so we can write

$$E = \{(r, \theta, z) | 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1, 1 - r^2 \leq z \leq 4\}$$

Since the density at (x, y, z) is proportional to the distance to the distance from the z-axis, the density function is

$$f(x, y, z) = K\sqrt{x^2 + y^2} = Kr$$

where K is the proportionality constant.

Therefore, the mass of E is

$$\begin{aligned} m = \int\int\int_E K\sqrt{x^2 + y^2}\, dV &= \int_0^{2\pi} \int_0^1 \int_{1-r^2}^4 (Kr)\, r\, dz\, dr\, d\theta \\ &= \int_0^{2\pi} \int_0^1 Kr^2 \left[4 - 1(1 - r^2)\right]\, dr\, d\theta \\ &= K \int_0^{2\pi} d\theta \int_0^1 (3r^2 + r^4)\, d\theta \\ &= 2\pi K \left[r^3 + \frac{r^5}{5}\right]_0^1 = \frac{12\pi K}{5} \end{aligned}$$