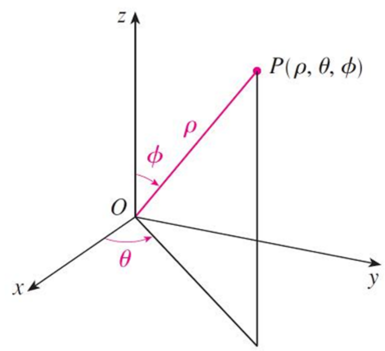
Triple Integrals in Spherical Coordinates

Another useful coordinate system in three dimensions is the spherical coordinate system.

The spherical coordinates $(\rho, \theta\phi)$ of a point P in space are shown in figure below, where $\rho = |OP|$ is the distance from the origin to P, θ is the same angle as in cylindrical coordinates, and ϕ is the angle between the positive z-axis and the line segment OP.

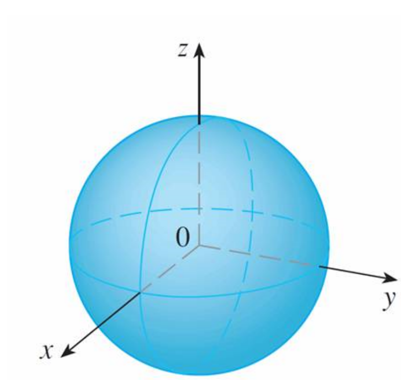


Note that

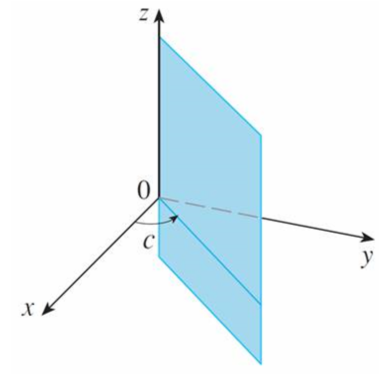
$$\rho \geq 0, \qquad 0 \leq \phi \leq \pi$$

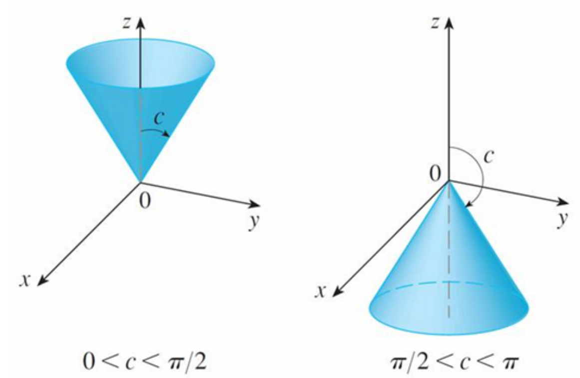
The spherical coordinate system is especially useful in problems where there is symmetry about a point, and the origin is placed at this point.

For example, the sphere with center the origin and radius c has the simple equation $\rho = c$ (see figure below); this is the reason for the name "spherical" coordinates.

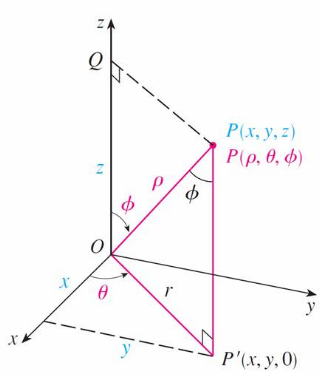


The graph of the equation $\theta = c$ is a vertical half-plane (See first figure below), and the equation $\phi = c$ represents a half-cone with the z-axis as its axis (see the second figure below with two cones).





The relationship between rectangular and spherical coordinates can be seen from the following figure.



From triangles OPQ and OPP',

$$z = \rho \cos \phi \qquad r = \rho \sin \phi$$

But $x = r\cos\theta$ and $y = r\sin\theta$, so to convert from spherical to rectangular coordinates, use

$$x = \rho \sin \phi \cos \theta \qquad y = \rho \sin \phi \sin \theta \qquad z = \rho \cos \phi \tag{1}$$

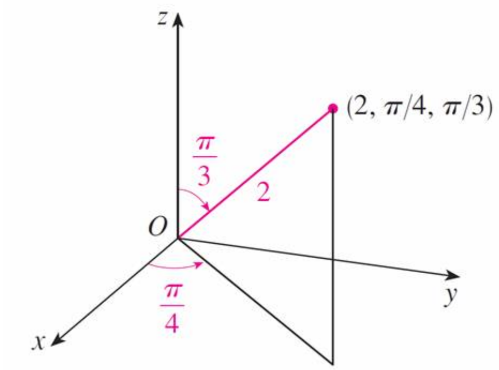
And the distance formula shows that

$$\rho^2 = x^2 + y^2 + z^2 \tag{2}$$

The formula is used to convert from rectangular to spherical coordinates.

Example 1. The point $(2, \pi/4, \pi/3)$ is given in spherical coordinates. Plot the point and find its rectangular coordinates.

Solution. The point can be plotted as in the following figure:



From Equation 1,

$$x = \rho \sin \phi \cos \theta = 2 \sin \frac{\pi}{3} \cos \frac{\pi}{4} = 2 \left(\frac{\sqrt{3}}{2} \right) \left(\frac{1}{\sqrt{2}} \right) = \sqrt{\frac{3}{2}}$$

$$y = \rho \sin \phi \sin \theta = 2 \sin \frac{\pi}{3} \sin \frac{\pi}{4} = 2 \left(\frac{\sqrt{3}}{2} \right) \left(\frac{1}{\sqrt{2}} \right) = \sqrt{\frac{3}{2}}$$

$$z = \rho \cos \phi = 2 \cos \frac{\pi}{3} = 2 \frac{1}{2} = 1$$

Thus, the point $(2, \pi/4, \pi/3)$ is $(\sqrt{3/2}, \sqrt{3/2}, 1)$ in rectangular coordinates.

Evaluating Triple Integrals with Spherical Coordinates

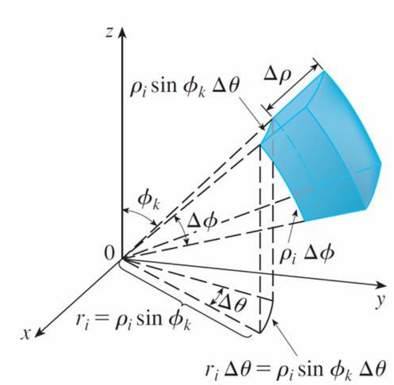
In the spherical coordinate system the counterpart of a rectangular box is a spherical wedge

$$E = \{(\rho, \theta, \phi) | a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\}$$

where $a \geq 0$ and $\beta - \alpha \leq 2\pi$ and $d - c \leq \pi$. Although we defined triple integrals by dividing solids into small boxes, it can be shown that dividing a solid into small spherical wedges always gives the same result.

So we divide E into smaller spherical wedges E_{ijk} by means of equally spaced spheres $\rho = \rho_i$, half-planes $\theta = \theta_j$, and half-cones $\phi = \phi_k$.

Figure below shows that E_{ijk} is approximately a rectangular box with dimensions $\Delta\rho$, $\rho_i \Delta\phi$ (arc of a circle with radius ρ_i, angle $\Delta\phi$), and $\rho_i \sin \phi_k \Delta\theta$ (arc of a circle with radius $\rho_i \sin \phi_k$, angle $\Delta\theta$).



So an approximation to the volume of E_{ijk} is given by

$$\Delta V_{ijk} \approxeq (\Delta\rho)(\rho_i \Delta\phi)(\rho_i \sin\phi_k \Delta\theta) = \rho_i^2 \sin\phi_k \Delta\rho\Delta\theta\Delta\phi$$

in fact, it can be shown, with the aid of the Mean Value Theorem, that the volume of E_{ijk} is given exactly by

$$\Delta V_{ijk} = \tilde{\rho}_i^2 \sin\tilde{\phi}_k \Delta\rho\Delta\theta\Delta\phi$$

where $(\tilde{\rho}_i, \tilde{\theta}_j, \tilde{\phi}_k)$ is some point in E_{ijk}.

Let $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ be the rectangular coordinates of this point. Then,

$$\int\int\int_F f(x,y,z)\,dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\,\Delta V_{ijk}$$

$$= \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(\tilde{\rho}_i \sin\tilde{\phi}_k \cos\theta_j, \tilde{\rho}_i \sin\tilde{\phi}_k \sin\tilde{\theta}_k, \tilde{\rho}_i \cos\tilde{\phi}_k)\,\tilde{\rho}_i^2 \sin\tilde{\phi}_k \Delta\rho\Delta\theta\Delta\phi$$

But this sum is a Riemann sum for the function

$$F(\rho, \theta\phi) = f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi$$

Consequently, we have arrived at the following formula for triple integration in spherical coordinates.

$$\int\int\int_E f(x,y,z)\,dV = \int_c^d\int_\alpha^\beta\int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi \qquad (3)$$

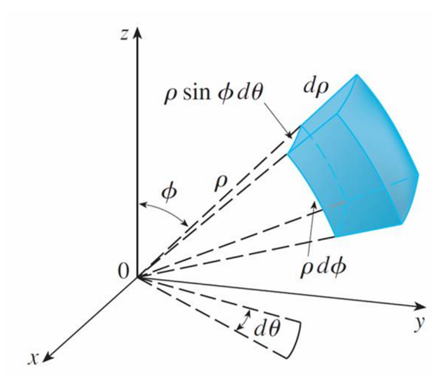
where E is a spherical wedge given by

$$E = \{(\rho.\theta,\phi)|a \le \rho \le b, \alpha \le \theta \le \beta, c \le \phi \le d\}$$

Formula 3 says that we convert a triple integral from rectangular coordinates to spherical coordinates by writing

$$x = \rho\sin\phi\cos\theta \qquad y = \rho\sin\phi\sin\theta \qquad z = \rho\cos\phi \qquad (4)$$

using the appropriate limits of integration, and replacing dv by $\rho^2\sin\phi\,dp\,d\theta\,d\phi$.

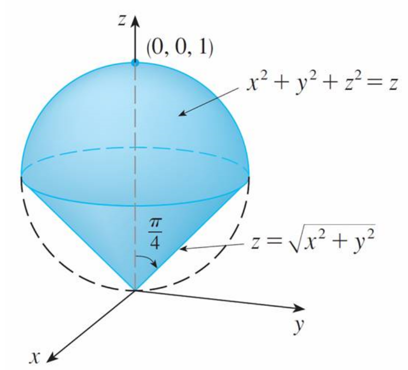


This formula can be extended to include more general spherical regions such as

$$E = \{(\rho,\theta,\phi)|\alpha \le \theta \le \beta, c \le \phi \le d\, g_1(\theta,\phi) \le \rho \le g_2(\theta,\phi)\}$$

In this case, the formula is the same as in 3 except that the limits of integration for ρ are $g_1(\theta, \phi)$ and $g_2(\theta, \phi)$.
Usually, spherical coordinates are used in triple integrals when surfaces such as cones and spheres form the boundary of the region of integration.

Example 2. Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$ as illustrated in the following figure.



Notice that the sphere passes through the origin and has center $(0, 0, \frac{1}{2})$. We write the equation of the sphere in spherical coordinates as

$$\rho^2 = \rho \cos \phi \qquad \text{or} \qquad \rho = \cos \phi$$

The equation of the cone can be written as

$$\begin{aligned} \rho \cos \phi &= \sqrt{\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta} \\ &= \rho \sin \phi \end{aligned}$$

This gives $\sin \phi = \cos \phi$, or $\phi = \pi/4$. Therefore the description of the solid E in spherical coordinates is

$$E = \{(\rho, \theta, \phi) | 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi/4 \, 0 \leq \rho \leq \cos \phi\}$$

Figures below shows how E is swept out if we integrate first with respect to ρ, then ϕ, and then θ.

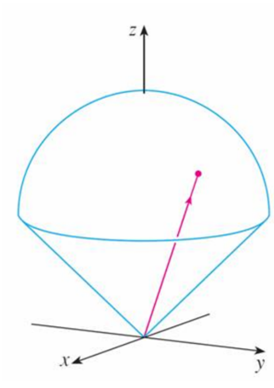


FIGURE 1. ρ varies from 0 to $cos\phi$ while ϕ and θ are constants.

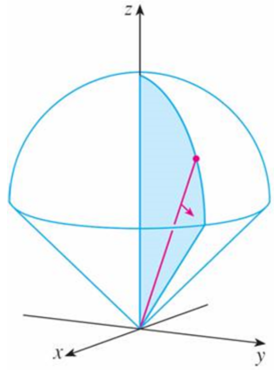


FIGURE 2. ϕ varies from 0 to $\pi/4$ while θ is constant.

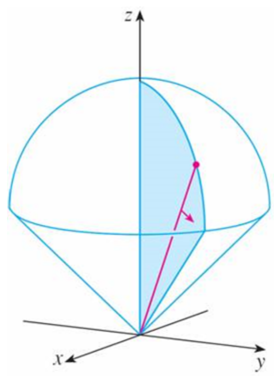


FIGURE 3. θ varies from 0 to 2π.

The volume of E is

$$
\begin{aligned}
V(E) &= \int\int\int_E dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi} \rho^2 \sin\phi \, dp \, d\phi \, d\theta \\
&= \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\phi \left[\frac{\rho^3}{3}\right]_{\rho=1}^{\rho=\cos\phi} d\phi \\
&= \frac{2\pi}{3}\int_0^{\pi/4} \sin\phi\cos^3\phi \, d\phi = \frac{2\pi}{3}\left[-\frac{\cos^4\phi}{4}\right]_0^{\pi/4} = \frac{\pi}{8}
\end{aligned}
$$