Change of Variables in Double Integrals

In one-dimensional calculus we often use a change of variable (a substitution) to simplify an integral. By reversing the roles of and , we can write the Substitution Rule (4.5.5 in textbook) as

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))g'(u)\,du \tag{1}$$

where $x = g(u)$ and $a = g(c)$,$b = g(d)$. Another way of writing Formula 1 is as follows:

$$\int_a^b f(x)\,dx = \int_c^d f(x(u))\frac{dx}{du}\,du \tag{2}$$

A change of variables can also be useful in double integrals. We have already seen one example of this: conversion to polar coordinates. The new variables r and θ are related to the old variables x and y by the equations

$$x = r\cos\theta \qquad y = r\sin\theta$$

and the change of variables formula (15.3.2 in textbook) can be written as

$$\int\int_R f(x,y)\,dA = \int\int_S f(r\cos\theta, r\sin\theta)r\,dr\,d\theta$$

where S is the region in the $r\theta$-plane that corresponds to the region R in the xy-plane.

More generally, we consider a change of variables that is given by a **transformation** T from the uv-plane to the xy-plane:

$$T(u,v) = (x,y)$$

where x and y are related to u and v by the equations

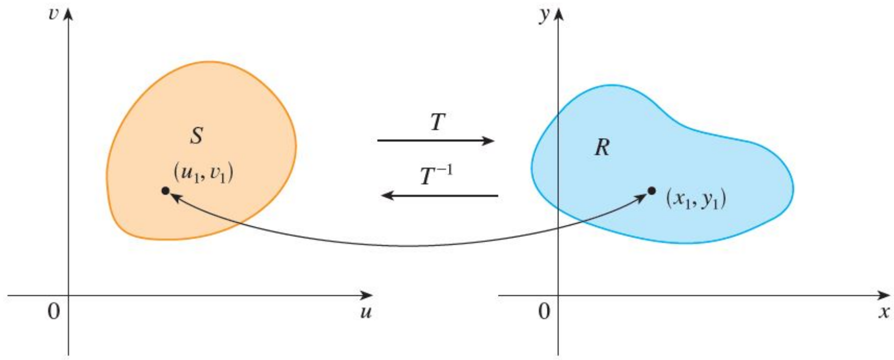
$$x = g(u,v) \qquad y = h(u,v) \tag{3}$$

or, as we sometimes write,

$$x = x(u,v) \qquad y = y(u,v)$$

We usually assume that T is a C^1 **transformation**, which means that g and h have continuous first-order partial derivatives.

A transformation T is really just a function whose domain and range are both subsets of $\mathbb{R}^2$. If $T = (u_1, v_1) = (x_1, y_1)$, then the point (x_1, y_1) is called the **image** of the point (u_1, v_1). If no two points have the same image,T is called **one-to-one**. Figure 1 shows the effect of a transformation T on a region S in the uv-plane.T transforms S into a region R in the xy-plane called the **image of S**, consisting of the images of all points in S.



If T is a one-to-one transformation, then it has an **inverse transformation** T^{-1} from the xy-plane to the uv-plane and it may be possible to solve Equations 3 for u and v in terms of x and y:

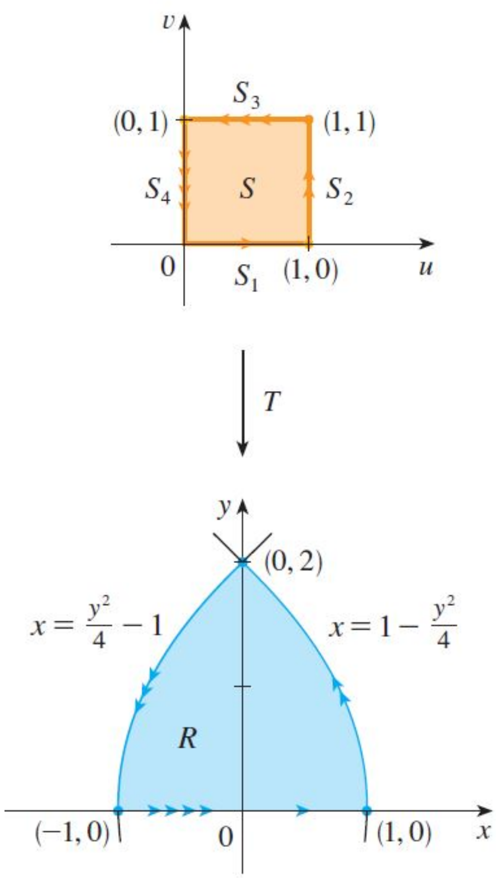
$$u = G(x, y) \qquad v = H(x, y)$$

Example 1. A transformation is defined by the equations

$$x = u^2 - v^2 \qquad y = 2uv$$

Find the image of the square $S = \{(u, v) | 0 \leqslant u \leqslant 1, 0 \leqslant v \leqslant 1\}$.

Solution. The transformation maps the boundary of S into the boundary of the image. So we begin by finding the images of the sides of S. The first side, S_1, is given by $v = 0 (0 \leqslant u \leqslant 1)$. See Figure below.)



From the given equations we have $x = u^2$,$y = 0$, and so $0 \leqslant x \leqslant 1$. Thus S_1 is mapped into the line segment from $(0, 0)$ to $(1, 0)$ in the xy-plane. The second side, S_2 is $u = 1(0 \leqslant v \leqslant 1)$ and, putting $u = 1$ in the given equations, we get

$$x = 1 - v^2 \qquad y = 2v$$

Eliminating v, we obtain

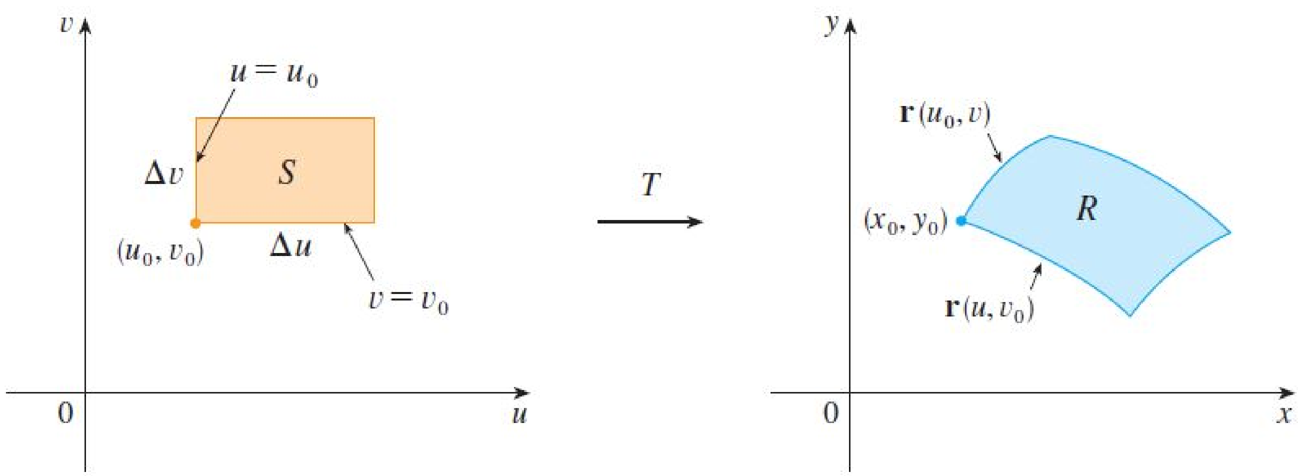
$$x = 1 - \frac{y^2}{4} \qquad 0 \leqslant x \leqslant 1 \tag{4}$$

which is part of a parabola. Similarly, S_3 is given by $v = 1(0 \leqslant u \leqslant 1)$, whose image is the parabolic arc

$$x = \frac{y^2}{4} - 1 \qquad -1 \leqslant x \leqslant 0 \tag{5}$$

Finally,S_4 is given by $u = 0(0 \leqslant v \leqslant 1)$ whose image is $x = -v^2$, $y = 0$, that is,$-1 \leqslant x \leqslant 0$. (Notice that as we move around the square in the counterclockwise direction, we also move around the parabolic region in the counterclockwise direction.) The image of S is the region R (shown in figure above) bounded by the x-axis and the parabolas given by Equations 4 and 5.

Now let's see how a change of variables affects a double integral. We start with a small rectangle S in the uv-plane whose lower left corner is the point (u_0, v_0) and whose dimensions are Δu and Δv.



The image of S is a region R in the xy-plane, one of whose boundary points is $(x_0, y_0) = T(u_0, v_0)$. The vector

$$\mathbf{r}(u, v) = g(u, v)\mathbf{i} + h(u, v)\mathbf{j}$$

is the position vector of the image of the point (u, v). The equation of the lower side of$S$ is $v = v_0$, whose image curve is given by the vector function $\mathbf{r}(u, v_0)$. The tangent vector at (x_0, y_0) to this image curve is

$$\mathbf{r}_u = g_u(u_0, v_0)\mathbf{i} + h_u(u_0, v_0)\mathbf{j} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j}$$

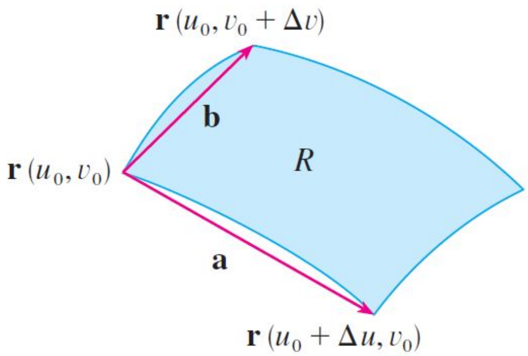
Similarly, the tangent vector at (x_0, y_0) to the image curve of the left side of S (namely,$u = u_0$) is

$$\mathbf{r}_v = g_v(u_0, v_0)\mathbf{i} + h_v(u_0, v_0)\mathbf{j} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j}$$

We can approximate the image region $R = T(S)$ by a parallelogram determined by the secant vectors

$$\mathbf{a} = \mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \qquad \mathbf{b} = \mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0)$$

shown in figure below.



But

$$r_u = \lim_{\Delta u \to 0} \frac{\mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0)}{\Delta u}$$

and so

$$\mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \approx \Delta u\, \mathbf{r}_u$$

Similarly

$$\mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0) \approx \Delta v\, \mathbf{r}_v$$

This means that we can approximate R by a parallelogram determined by the vectors $\Delta u\, \mathbf{r}_u$ and $\Delta v\, \mathbf{r}_v$. (See Figure 5.) Therefore we can approximate the area of R by the area of this parallelogram, which, from Section 12.4, is

$$|(\Delta u\, \mathbf{r}_u) \times (\Delta v\, \mathbf{r}_v)| = |\mathbf{r}_u \times \mathbf{r}_v|\, \Delta u\, \Delta v \tag{6}$$

Computing the cross product, we obtain

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \mathbf{k} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \mathbf{k}$$

The determinant that arises in this calculation is called the *Jacobian* of the transformation and is given a special notation.

Definition 1. The **Jacobian** of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

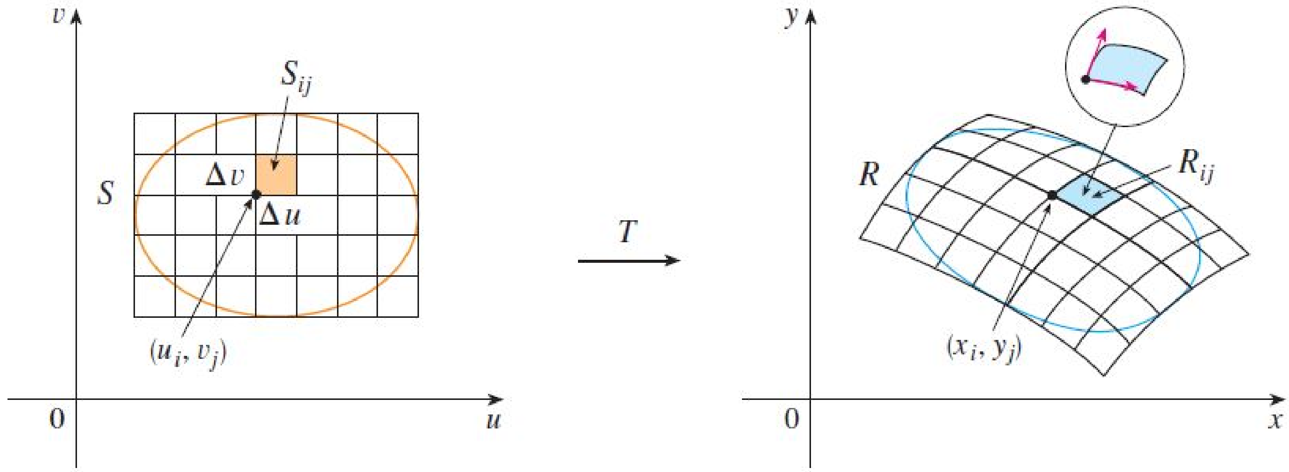
$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

With this notation we can use Equation 6 to give an approximation to the area ΔA of R:

$$\Delta A \approx \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u \Delta v \tag{7}$$

where the Jacobian is evaluated at (u_0, v_0).

Next we divide a region S in the uv-plane into rectangles S_{ij} and call their images in the xy-plane R_{ij}. (See figure below)



Applying the approximation (8) to each R_{ij} we approximate the double integral of f over R as follows:

$$\begin{aligned}\int\int_R f(x,y)\,dA &\approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i,y_j)\Delta A \\ &\approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(g(u_i,v_j),h(u_i,v_j))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\Delta u\,\Delta v\end{aligned}$$

where the Jacobian is evaluated at (u_i, v_j). Notice that this double sum is a Riemann sum for the integral

$$\int\int_S f(g(u,v),h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,du\,dv$$

The foregoing argument suggests that the following theorem is true.

Theorem 1. *Suppose that T is a C^1 transformation whose Jacobian is nonzero and that maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then*

$$\int\int_R f(x,y)\,dA = \int\int_S f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,du\,dv \tag{8}$$

Theorem 1 says that we change from an integral in x and y to an integral in u and v by expressing x and y in terms of u and v and writing

$$dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,du\,dv$$

Notice the similarity between Theorem 9 and the one-dimensional formula in Equation **??**. Instead of the derivative $\dfrac{dx}{du}$, we have the absolute value of the Jacobian, that is, $\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right|$.

As a first illustration of Theorem 1, we show that the formula for integration in polar coordinates is just a special case. Here the transformation T from the $r\theta$-plane to the xy-plane is given by

$$x = g(r, \theta) = r\cos\theta \qquad y = h(r, \theta) = r\sin\theta$$

and the geometry of the transformation is shown in Figure 7 below.

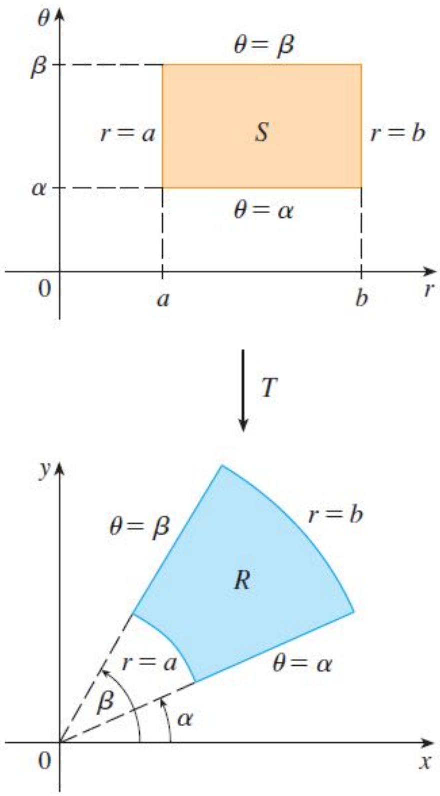


FIGURE 7
The polar coordinate transformation

T maps an ordinary rectangle in the $r\theta$-plane to a polar rectangle in the xy-plane. The Jacobian of T is

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r > 0$$

Thus Theorem 1 gives

$$\begin{aligned} \int\int_R f(x, y)\,dx\,dy &= \int\int_S f(r\cos\theta, r\sin\theta) \left|\frac{\partial(x, y)}{\partial(r, \theta)}\right| dr\,d\theta \\ &= \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta \end{aligned}$$

which is the same as Formula 15.3.2 in our textbook.

Example 2. Use the change of variables $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral $\int\int_R y\,dA$, where R is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, $y \geqslant 0$.

Solution. The region R is pictured in Figure 2 (on page 1094). In Example 1 we discovered that $T(S) = R$, where S is the square $[0,1] \times [0,1]$. Indeed, the reason for making the change of variables to evaluate the integral is that S is a much simpler region than R. First we need to compute the Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2 > 0$$

Therefore, by Theorem 1,

$$\begin{aligned} \int\int_R y\,dA &= \int\int_S 2uv \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| dA = \int_0^1 \int_0^1 (2uv)4(u^2+v^2)\,du\,dv \\ &= 8\int_0^1 \int_0^1 (u^3v + uv^3)\,du\,dv = 8\int_0^1 \left[\frac{1}{4}u^4v + \frac{1}{2}u^2v^3\right]_{u=0}^{u=1} dv \\ &= \int_0^1 (2v + 4v^3)\,dv = \left[v^2 + v^4\right]_0^1 = 2 \end{aligned}$$

Note: Example 2 was not a very difficult problem to solve because we were given a suitable change of variables. If we are not supplied with a transformation, then the first step is to think of an appropriate change of variables. If $f(x,y)$ is difficult to integrate, then the form of $f(x,y)$ may suggest a transformation. If the region of integration R is awkward, then the transformation should be chosen so that the corresponding S region in the uv-plane has a convenient description.

Example 3. Evaluate the integral $\int\int_R e^{(x+y)/(x-y)}\,dA$, where R is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,-2)$, and $(0.-1)$.

Solution. Since it isn't easy to integrate $e^{(x+y)/(x-y)}$, we make a change of variables suggested by the form of this function:

$$u = x + y \qquad v = x - y \tag{9}$$

These equations define a transformation T^{-1} from the xy-plane to the uv-plane. Theorem 9 talks about a transformation T from the uv-plane to the xy-plane. It is obtained by solving Equations 10 for x and y:

$$x = \frac{1}{2}(u+v) \qquad y = \frac{1}{2}(u-v) \tag{10}$$

The Jacobian of T is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -1/2$$

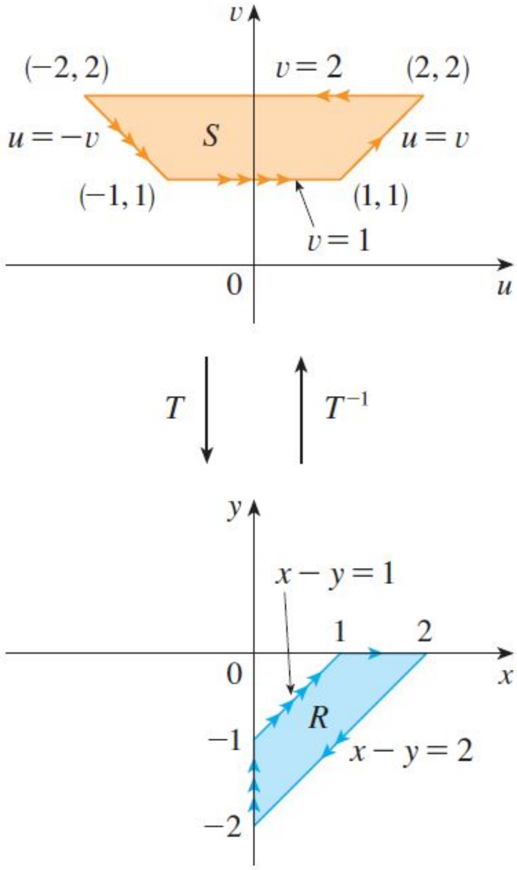
To find the region S in the uv-plane corresponding to R, we note that the sides of R lie on the lines

$$y = 0 \qquad x - y = 2 \qquad x = 0 \qquad x - y = 1$$

and, from either Equations 10 or Equations 11, the image lines in the uv-plane are

$$u = v \qquad v = 2 \qquad u = -v \qquad v = 1$$

Thus the region S is the trapezoidal region with vertices $(1,1)$, $(2,2)$, $(-2,2)$, and $(-1,1)$ shown in figure below.



Since

$$S = \{(u, v) | 1 \leqslant v \leqslant 2, -v \leqslant u \leqslant v\}$$

Theorem 1 gives

$$\int\int_R e^{(x+y)/(x-y)}\, dA = \int\int_S e^{u/v} \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

$$= \int_1^2 \int_{-v}^{v} e^{u/v} \left(\frac{1}{2}\right) du\, dv = \frac{1}{2} \left[ve^{u/v}\right]_{u=-v}^{u=v} dv$$

$$= \frac{1}{2} \int_1^2 (e - e^{-1}) v\, dv = \frac{3}{4}(e - e^{-1})$$