1. Find and sketch the **domain** of the following functions (At least make sure you can see the region).

(a) $f(x, y) = \sqrt{x^2 - y}$.

Solution. $D = \{(x, y)|x^2 - y \geq 0\} = \{(x, y)|x^2 \geq y\}$.

(b) $f(x, y) = \ln(-x^2 - y^2 + 2)$.

Solution. $D = \{(x, y)|(-x^2 - y^2 + 2) > 0\} = \{(x, y)|x^2 + y^2 < 2\}$.

Notice that we obtain the domain in the form of equation for circle. We conclude that the domain is the region bounded by a circle with radius 2 and center at origin.

(c) $f(x, y) = \dfrac{1}{x} + \sqrt{y - 1} + \sqrt{x + 1}$.

Solution. Notice that the function is always defined when $\frac{1}{x}$, $\sqrt{y-1}$, and $\sqrt{x+1}$ is defined. So, $D = \{(x, y)|x \neq 0, y - 1 \geq 0, x + 1 \geq 0\} = \{(x, y)|x \neq 0, y \geq 1, x \geq -1\}$.

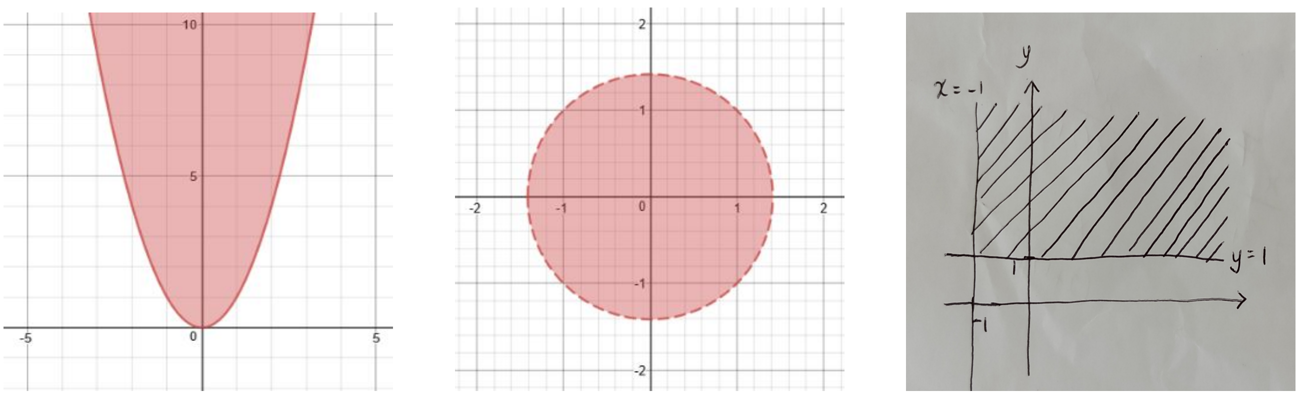

FIGURE 1. The graphs from left to right are the sketches for Questions (a), (b) and (c), respectively

**Please make sure the interceptions with the axes are clearly labelled. For Question (c), the line $x = 0$ is excluded from the domain.

2. Identify and sketch the **contour plot** for the following functions.

(a) $2x - y + z^2 = 0$ for $k = 0, 1, 2, 3$.

Solution. By substituting k, we have $2x - y + k^2 = 0$ or $y = 2x + k^2$. Using the values of k given, we can see the graph as in the first figure below.

(b) $x - y^2 + z = 1$ for $k = 0, 1, 2, 3$.

Solution. By substituting k, we have $x - y^2 + k = 1$ or $x = y^2 - k + 1$ which is a quadratic equation. Using the values of k given, we can see the graph as in the second figure below.

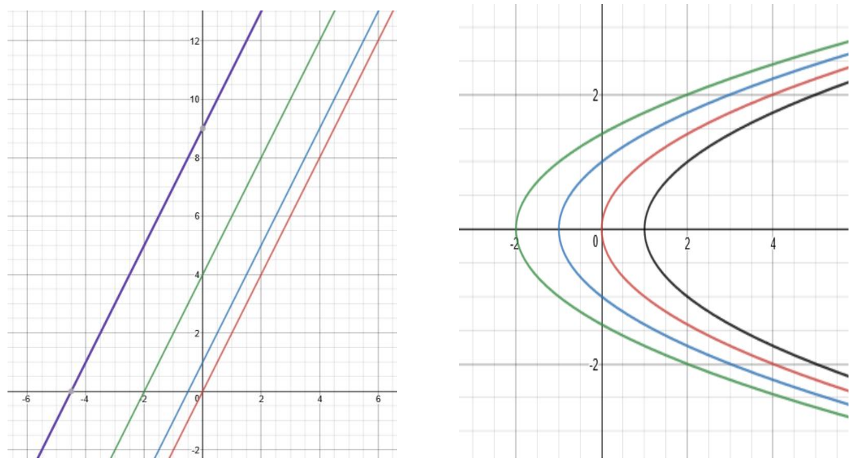

FIGURE 2. The graphs from left to right are the sketches for Questions (a) and (b) respectively

**For the first figure the lines from left to right is for values $k = 3, 2, 1, 0$ respectively.

**For the second figure the lines from the outermost to the innermost is for values $k = 3, 2, 1, 0$ respectively.

3. Evaluate the following **limit**.

(a) $\lim\limits_{(x,y)\to(1,2)} \dfrac{3x-2y}{x+y}$.

Solution. We can directly substitute $x = 1$ and $y = 2$ into the equation since it is a rational function. The limit exist everywhere as long as $x + y \neq 0$. Thus, $\lim\limits_{(x,y)\to(1,2)} \dfrac{3x-2y}{x+y} = \dfrac{-1}{3}$.

(b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$.

Solution. First, we shall compute the limit as $(x, y) \to (0, 0)$ along x-axis i.e. $y = 0$ and along y-axis i.e. $x = 0$. We have $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2} = \lim\limits_{(x,y)\to(0,0)} \dfrac{x^2}{x^2} = 1$.

Then, we consider along y-axis i.e. $x = 0$. We have

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{-y^2}{y^2} = -1.$$

Since the limits are different, we can conclude that the limit does not exist (DNE).

(c) $\lim\limits_{(x,y)\to(1,3)} \dfrac{3x^3-yx^2}{9x^2-y^2}$.

Solution. $\lim\limits_{(x,y)\to(1,3)} \dfrac{3x^3-x^2y}{9x^2-y^2} = \lim\limits_{(x,y)\to(1,3)} \dfrac{x^2(3x-y)}{(3x-y)(3x+y)} = \lim\limits_{(x,y)\to(1,3)} \dfrac{x^2}{3x+y} = \dfrac{1}{6}$

(d) $\lim_{(x,y)\to(0,0)} \frac{x+y}{4x+2y}$.

Solution. This is similar to Equation (b). Please compute the limit as (x, y) approaches $(0, 0)$ from several paths. You will see that the limit DNE.

(e) $\lim_{(x,y)\to(0,0)} \frac{xy^2}{2x+y}$.

Solution. First we check that the limit from various path such as when $x = 0$, $y = 0$, or $y = x^2$ and all the limit is equal to 0. But, try approaches $(0, 0)$ along the line $y = mx$, you will see that the limit blows up to ∞.

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{xy^2}{2x+y} &= \lim_{(x,y)\to(0,0)} \frac{x(mx)^2}{2x+mx} \\
&= \lim_{(x,y)\to(0,0)} \frac{m^2x^3}{x(2+m)} \\
&= \lim_{(x,y)\to(0,0)} \frac{m^2x}{2+m}
\end{aligned}$$

The idea behind substituting $y = mx$ as both variables approach zero is that the limit shouldn't change whenever you change the value of m. It is not the other way round where you are free to choose any m to make the limit equal to zero. Let say you choose $y = -2x$ where $m = 2$.

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{xy^2}{2x+y} &= \lim_{(x,y)\to(0,0)} \frac{(-2)^2x}{2-2} \\
&= \lim_{(x,y)\to(0,0)} \frac{4x}{0}
\end{aligned}$$

(f) $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$. (*Hint: Find a relevant $g(x)$ and use Squeeze Theorem*)

Solution. You shall check that the limit from various directions or paths e.g. $x = 0$, $y = 0$ gives 0. So, you might suspect that the limit exist. To use Squeeze Theorem, we need to find a function $g(x, y)$ (similar to $g(x)$ in single variable function) such that $|f(x, y) - 0| \leq g(x, y)$ and $\lim_{(x,y)\to(0,0)} = 0$.

To find g,

$$\begin{aligned}
|f(x,y) - 0| &= \left|\frac{x^2y}{x^2+y^2} - 0\right| \\
&= \left|\frac{x^2y}{x^2+y^2}\right| \\
&= \frac{x^2|y|}{x^2+y^2}
\end{aligned}$$

Since we can make a fraction bigger by making its denominator smaller,

$$\begin{aligned}|f(x,y)-0| &= \frac{x^2|y|}{x^2+y^2}\\ &\leq \frac{x^2|y|}{x^2}\\ &\leq |y|\end{aligned}$$

If we let $g(x,y)=|y|$, we see $\lim_{(x,y)\to(0,0)} g(x,y)=0$. Thus, $\lim_{(x,y)\to(0,0)} f(x,y)=0$.

4. Find the region where the following functions are **continuous**.

(a) $f(x,y)=\dfrac{1}{x^2-y}$.

Solution. Since $f(x)$ is a rational function (quotient of two polynomials at both denominator and numerator), then it is continuous on its domain. Therefore it is continuous as long as the denominator is not 0 i.e. $x^2-y\neq 0$. Therefore, f is continuous on $\{(x,y)\in\mathbb{R}^2 | x^2\neq y\}$

(b) $f(x,y)=\arctan\left(\dfrac{xy^2}{x+y}\right)$.

Solution. The given function is a composition of two functions. It is known that arctan is continuous on its domain, $\mathbb{R}$. Therefore, the function is continuous when $\dfrac{xy^2}{x+y}$ is continuous. With steps analogous to Question 4(a), we say that $f(x,y)=\dfrac{1}{x^2-y}$ is continuous on $\{(x,y)\in\mathbb{R}^2 | x^2\neq y\}$.

(c) $f(x,y)=\ln(x^2+y^2-1)$.

Solution. Similarly, we know than a natural logarithm is always continuous whenever its argument is greater than 0. Then, the function is continuous whenever $x^2+y^2-1>0$ or $x^2+y^2>1$. If we observe, we know that $x^2+y^2>1$ is the region outside a circle of radius 1. Thus, we can conclude that $f(x,y)=\ln(x^2+y^2-1)$ is continuous on the portion of $\mathbb{R}^2$ outside the circle with radius 1, and the origin as its center or, if you wish $\{(x,y)\in\mathbb{R}^2 | x^2+y^2>1\}$.

5. Determine the region of **continuity** for the following function.

$$f(x,y)=\begin{cases}\dfrac{x^2y}{x^2+y^2} & \text{if } (x,y)\neq(0,0),\\ 0, & \text{if } (x,y)=(0,0).\end{cases}$$

Solution. First, you have to show that $\lim_{(x,y)\to(0,0)} \dfrac{x^2y}{x^2+y^2}=0$. You may use the $\epsilon-\delta$ technique.

Let $\epsilon>0$. We want to find $\delta>0$ such that

$$if \qquad 0<\sqrt{x^2+y^2}<\delta \qquad \text{then} \qquad \left|\frac{x^2y}{x^2+y^2}-0\right|<\epsilon$$

that is

$$if \qquad 0 < \sqrt{x^2 + y^2} < \delta \qquad \text{then} \qquad \frac{x^2|y|}{x^2 + y^2} < \epsilon$$

But $x^2 \leq x^2 + y^2$ since $y^2 \geq 0$, so $x^2/(x^2 + y^2) \leq 1$, then,

$$\frac{x^2|y|}{x^2 + y^2} \leq |y| = \sqrt{y^2} \leq \sqrt{x^2 + y^2}.$$

Thus, if we choose $\delta = \epsilon$ and let $0 < \sqrt{x^2 + y^2} < \delta$, then

$$\left| \frac{x^2 y}{x^2 + y^2} - 0 \right| \leq \sqrt{x^2 + y^2} < \delta = \epsilon.$$

Hence,

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2 + y^2} = 0.$$

Therefore, f is continuous at $(0, 0)$. It follows that f is continuous everywhere.