1. For first question, the question is very straightforward. You have to find all first partial derivatives of the given function with respect to each independent variable. To do so, treat other variable(s) as constant(s).

(a) $f(x,y) = \cos(2xy) + xe^{-x}$

Solution.

$$\begin{aligned} f_x &= -\sin(2xy).2y + e^{-x} - xe^{-x} \\ &= -2y\sin(2xy) + e^{-x}(1-x) \\ f_y &= -2x\sin(2xy) \end{aligned}$$

Note: Product rule and Chain Rule from MAT101 is used to find f_x.

(b) $f(x,y) = \dfrac{x^2}{\cos x} + 2x\cos(2y)$

Solution.

$$\begin{aligned} f_x &= \frac{2x\cos(x) + x^2\sin(x)}{\cos^2(x)} + 2\cos(2y) \\ f_y &= 2x(-2\sin(2y)) \\ &= -4x\sin(2y) \end{aligned}$$

Note: Use Quotient Rule to find f_x.

(c) $f(x,y) = \dfrac{4x^3 + 2xy}{2\tan x} = \dfrac{1}{2} \cdot \dfrac{4x^3 + 2xy}{\tan x}$

Solution.

$$\begin{aligned} f_x &= \frac{(12x^2 + 2y)\tan(x) - (4x^3 + 2xy)\sec^2(X)}{2\tan^2(x)} \\ f_y &= \frac{2x}{2\tan x} \end{aligned}$$

Note: Use Quotient Rule to find f_x.

(d) $f(x,y,z) = 3xy + 2yz - \dfrac{xz}{2}$

Solution.

$$\begin{aligned} f_x &= 3y - \frac{z}{2} \\ f_y &= 3x + 2z \\ f_z &= 2y - \frac{x}{2} \end{aligned}$$

(e) $f(x,y,z) = \cos(xyz) + x^2$

Solution. (Consider Chain Rule from MAT101)

$$f_x = -\sin(xyz).yz + 2x = -yz\sin(xyz) + 2x$$
$$f_y = -xz\sin(xyz)$$
$$f_z = -xy\sin(xyz)$$

(f) $f(x, y, z) = \dfrac{x^2}{y} + 5x\sin(2yz^2)$

Solution.

$$f_x = \frac{2x}{y} + 5\sin(2yz^2)$$
$$f_y = -\frac{x^2}{y^2} + 10xz^2\cos(2yz^2)$$
$$f_z = 5x\cos(2yz^2).2y.2z = 20xyz\cos(2yz^2)$$

2. Find all second partial derivatives for the following functions.

(a) $f(x, y) = \dfrac{4x^3 + 2xy}{2\tan y}$

Solution.

$$f_x = \frac{12x + 2y}{2\tan(y)} = \frac{6x + y}{\tan(y)}$$
$$f_{xx} = \frac{6}{\tan(y)}$$
$$f_{xy} = 6x\frac{\partial}{\partial y}\left(\frac{1}{\tan(y)}\right) + \frac{\partial}{\partial y}\left(\frac{y}{\tan(y)}\right) \quad \text{(Use Quotient Rule on second term)}$$
$$= -\frac{6x\sec 2(y)}{\tan^2(y)} + \frac{\tan(y) - y\sec^2(y)}{\tan^2(y)}$$

$$f_y = -2x^3\frac{\sec^2(y)}{\tan^2(y)} + \frac{x(\tan(y) - y\sec^2(y))}{\tan^2(y)} \quad \text{(Use Quotient Rule)}$$
$$f_{yx} = -6x^2\frac{\sec^2(y)}{\tan^2(y)} + \frac{(\tan(y) - y\sec^2(y))}{\tan^2(y)}$$
$$f_{yy} = -2x^3\frac{2\sec^2(y)\tan^3(y) - 2\sec^4(y)\tan(y)}{\tan^4(y)}$$
$$+ \frac{2y\sec^2(y)\tan^3(y) - 2\sec^2(y)\tan(y)(tan(y) - y\sec^2(y)}{\tan^4(y)}$$

Note: Quotient rule is used a lot in Question 2(a). You must also recall the derivative for tan and sec. If case you don't remember $\tan x = \dfrac{\sin x}{\cos x}$.

(b) $f(x, y, z) = \dfrac{x^2}{y} + 5x\sin(2yz^2)$

Solution.

$$f_x = \frac{2x}{y} + 5\sin(2yz^2),$$
$$f_{xx} = \frac{2}{y}$$
$$f_{xy} = -\frac{2x}{y^2} + 10z^2\cos(2yz^2)$$
$$f_{xz} = 20yz\cos(2yz^2)$$

$$f_y = -\frac{x^2}{y^2} + 10xz^2\cos(2yz^2)$$
$$f_{yx} = -\frac{2x}{y^2} + 10z^2\cos(2yz^2)$$
$$f_{yy} = \frac{2x^2}{y^3} + 10xz^2(-\sin(2yz^2).2z^2)$$
$$= \frac{2x^2}{y^3} - 20xz^4\sin(2yz^2)$$
$$f_{yz} = -10xz^2\sin(2yz^2).4yz + 20xz\cos(2yz^2)$$
$$= -40xyz^3\sin(2yz^2) + 20xz\cos(2yz^2)$$

3. We will use the equation for the tangent of the plane, $z - z_0 = f_x(a,b)(x - x_0) + f_y(a,b)(y - y_0)$ to solve these problems.

(a) $f(x,y) = 3y^2 + 2xy + \dfrac{1}{y}$ at $(2,3,4)$

Solution.

$f_x(x,y) = 2y \qquad f_x(2,3) = 6$

$f_y(x,y) = 6y + 2x - \dfrac{1}{y^2} \qquad f_y(2,3) = 21\dfrac{8}{9}$

So,

$$z - 4 = f_x(2,3)(x-2) + f_y(2,3)(y-3)$$
$$z - 4 = 6(x-2) + 21\frac{8}{9}(y-3)$$
$$z = 6x + 21\frac{8y}{3} - 12 - 21\frac{8}{3} + 4$$
$$z = 6x + 21\frac{8y}{3} - 26\frac{1}{3}$$

Note. Observe that the point is not on the graph. This question is construct to test your application of the formula. The valid point will be $(2, 3, 39\frac{1}{3})$

(b) $f(x,y) = x^3 - y^3 - 32$ at $(1,2,3)$

Solution.

$f_x(x, y) = 3x^2 \qquad f_x(1, 2) = 3$
$f_y(x, y) = -3y^2 \qquad f_y(1, 2) = -12$

So,

$$z - 3 = f_x(1, 2)(x - 2) + f_y(1, 2)(y - 2)$$
$$z - 3 = 3(x - 2) - 12(y - 2)$$
$$z = 3x - 12y + 21$$

Note. Observe that the point is not on the graph. The valid point will be $(1, 2, -39)$.

(c) $f(x, y) = 2 + (x - 1)^2 + (2 + y)^2$ at $(2, 1, 3)$
Solution.

$f_x(x, y) = 2(x - 1) \qquad f_x(2, 1) = 2$
$f_y(x, y) = 2(2 + y) \qquad f_y(2, 1) = 6$

So,

$$z - 3 = f_x(2, 1)(x - 2) + f_y(2, 1)(y - 1)$$
$$z - 3 = 2(x - 2) + 6(y - 1)$$
$$z = 2x + 6y - 7$$

Note. Observe that the point is not on the graph. The valid point will be $(2, 1, 4)$.

Note. *For Question 4 and 5, we will use the linearization formula at* (a, b) *given by*

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

and the linear approximation formula at (a, b) *given by*

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

4. Find linearization of the function of the function $f(x, y) = 4x^3y^2 - \ln y$ at $(3, 1)$ in the domain and use it to approximate $f(3.01, 0.99)$.

Solution.

$f_x(x, y) = 12x^2y^2, \qquad f_x(3, 1) = 108$
$f_y(x, y) = 8x^3y, \qquad f_y(3, 1) = 216$
$f(3, 1) = 108$

$$L(x, y) = f(3, 1) + f_x(3, 1)(x - 3) + f_y(3, 1)(y - 1)$$
$$= 108 + 108(x - 3) + 216(y - 1)$$

Now, we will approximate the function at $(3.01, 0.99)$.

$$
\begin{aligned}
f(x,y) &\approx 108 + 108(x-3) + 216(y-1) \\
&\approx 108 + 108(3.01-3) + 216(0.99-1) \\
&\approx 106.92
\end{aligned}
$$

The actual value is $f(3.01, 0.99) = 4(3.01)^3(0.99)^2 - \ln(0.99) = 106.923$, similar to the approximate value up to two decimal points.

5. Find linearization of the function of the function $f(x,y) = 2xy^2e^{3y}$ at $(1,1)$ in the domain and use it to approximate $f(0.99, 0.99)$ and $f(1.50, 0.50)$. Compare the values obtained with the actual values from $f(x,y)$. What can you say?

Solution.

$$
\begin{aligned}
&f_x(x,y) = 2y^3e^{3y}, \qquad f_x(1,1) = 2e^3 \\
&f_y(x,y) = 2x(y^3.3e^{3y} + 3y^2e^{3y}), \qquad f_y(1,1) = 12e^3 \\
&f(3,1) = 2(1)(1^2)3^{3(1)} = 2e^3
\end{aligned}
$$

$$
\begin{aligned}
L(x,y) &= f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1) \\
&= 2e^3 + 2e^3(x-1) + 12e^3(y-1)
\end{aligned}
$$

Now, we will approximate the function at $(0.99, 0.99)$.

$$
\begin{aligned}
f(x,y) &\approx 2e^3 + 2e^3(0.99-1) + 12e^3(0.99-1) \\
&\approx 37.3591
\end{aligned}
$$

For actual values of at $(0.99, 0.99)$, we have $f(0.99, 0.99) = 2(0.99)(0.99^3)e^{3(0.99)} \approx 37.4477$. We can conclude that the approximation is close to the actual value.

Next, we approximate the values at $(1.5, 0.5)$.

$$
\begin{aligned}
f(x,y) &\approx 2e^3 + 2e^3(1.5-1) + 12e^3(0.5-1) \\
&\approx -60.2566
\end{aligned}
$$

For actual values of at $(0.99, 0.99)$, we have $f(1.5, 0.5) = 2(1.5)(0.5^3)e^{3(0.5)} \approx 1.6806$. We can conclude that the approximation is away from the actual value. This is due to the fact that the linear approximation only works when the point considered is close to the point where we obtain the linearization.

6. For question 6, we will use the formula $df = f_x(x,y,z)dx + f_y(x,y,z)dy + f_z(x,y,z)$.

(a) $f(x,w) = we^{x/w}$

Solution.

$$
\begin{aligned}
f_x(x,w) &= we^{x/w}.\frac{1}{w} \\
f_w(x,w) &= we^{x/w}.\frac{-x}{w^2} + e^{x/w}
\end{aligned}
$$

Then, $df = e^{x/w}dx + \left(e^{x/w} - \frac{xe^{x/w}}{w}\right) dw$

(b) $f(x, y, z) = x^3 - ye^{4z} + 5$

Solution.
$f_x(x, y, z) = 3x^2$
$f_y(x, y, z) = -e^{4z}$
$f_z(x, y, z) = -4ye^{4z}$

Then, $df = 3x^2dx - e^{4z}dy - 4ye^{4z}dz$

(c) $h(x, z, s, t) = sx^{-1} + 2zt^{-1} + 3xs^{-1} + 4tz^{-1}$

Solution.
$h_x(x, z, s, t) = -sx^{-2} + 3s^{-1}$
$h_z(x, z, s, t) = 2t^{-1} - 4tz^{-2}$
$h_s(x, z, s, t) = x^{-1} - 3xs^{-2}$
$h_t(x, z, s, t) = -2zt^{-2} + 4z^{-1}$

Then, $df = (-sx^{-2} + 3s^{-1})dx + (2t^{-1} - 4tz^{-2})dz + (x^{-1} - 3xs^{-2})ds + (-2zt^{-2} + 4z^{-1})dt$

7. Consider your answer in 6(a). If x changes from 3 to 2.95 and w changes from 2.5 to 2.55, compare the values of dz and Δz.

$$df = e^{3/2.5}(2.95 - 3) + \left(e^{3/2.5} - \frac{3e^{3/2.5}}{2.5}\right)(2.55 - 2.5) \approx -0.1992$$

Meanwhile,

$$\begin{aligned}\Delta f &= f(x, w) - f(x_0, w_0)\\ &= 2.55e^{2.95/2.55} - 2.5e^{3/2.5}\\ &\approx -0.19144\end{aligned}$$

Can you conclude on this?