1. Use chain rule to find $\dfrac{df}{dt}$ for the following functions.

Recall the formula for Chain Rule of the first case given by:

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

(a) $f(x,y) = \sqrt{x^2+y^2}$, $x = t$, $y = t^2$.

Solution. First we compute $\dfrac{\partial f}{\partial x}, \dfrac{dx}{dt}, \dfrac{\partial f}{\partial y}$ and $\dfrac{dy}{dt}$. We have,

$$\frac{\partial f}{\partial x} = \frac{1}{2}(x^2+y^2)^{-1/2}.2x = \frac{x}{\sqrt{x^2+y^2}}, \quad \frac{dx}{dt} = 1, \quad \frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}, \quad \frac{dy}{dt} = 2t$$

Then,

$$\frac{df}{dt} = \frac{x}{\sqrt{x^2+y^2}} + \frac{y}{\sqrt{x^2+y^2}}.2t$$

(b) $f(x,y) = ln(x+y)$, $x = e^t$, $y = e^t$.

Solution. Similar to Question (a),

$$\frac{\partial f}{\partial x} = \frac{1}{x+y}, \quad \frac{dx}{dt} = e^t, \quad \frac{\partial f}{\partial y} = \frac{1}{x+y}, \quad \frac{dy}{dt} = e^t$$

Then,

$$\frac{df}{dt} = \frac{e^t}{x+y} + \frac{e^t}{x+y} = \frac{2e^t}{x+y}$$

(c) $f(x,y,z) = x^2+y^3+z^4$ along the curve $\mathbf{r}(t) = \langle \cos(t), \sin(t), 3t\rangle$.

Solution. Since this is a function of three variables, then,

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

So, we have

$$\frac{df}{dt} = (2x)(-\sin(t)) + (3y^2)(\cos(t)) + (4z^3)(3)$$

2. Find $\dfrac{\partial f}{\partial s}$ and $\dfrac{\partial f}{\partial t}$ for the following functions. For this question we shall use the formula of Chain Rule for Case 2.

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s},$$

and

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}.$$

(a) $f(r, \theta) = e^r \cos\theta$, $r = st$, $\theta = \sqrt{s^2+t^2}$.

Solution.

$$\frac{\partial f}{\partial r} = e^r \cos\theta, \quad \frac{\partial r}{\partial s} = t, \quad \frac{\partial f}{\partial \theta} = -e^r \sin\theta, \quad \frac{\partial \theta}{\partial s} = \frac{s}{\sqrt{s^2+t^2}}$$

Then,

$$\frac{\partial f}{\partial s} = e^r \cos\theta . t - e^r \sin\theta \frac{s}{\sqrt{s^2+t^2}} = e^r \left(t\cos\theta - \frac{s\sin\theta}{\sqrt{s^2+t^2}} \right)$$

For $\frac{\partial f}{\partial t}$, the partial derivatives $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$ can be used from the earlier part.

$$\frac{\partial r}{\partial t} = s, \quad \frac{\partial \theta}{\partial t} = \frac{t}{\sqrt{s^2+t^2}}$$

Then,

$$\frac{\partial f}{\partial t} = e^r \cos\theta . s - e^r \sin\theta \frac{t}{\sqrt{s^2+t^2}} = e^r \left(s\cos\theta - \frac{t\sin\theta}{\sqrt{s^2+t^2}} \right)$$

(b) $f(x, y) = e^{xy^2}$, $x = s - t^2$, $y = t - s^2$.

Solution.

$$\frac{\partial f}{\partial x} = y^2 e^{xy^2}, \quad \frac{\partial x}{\partial s} = 1, \quad \frac{\partial f}{\partial y} = 2xye^{xy^2}, \quad \frac{\partial y}{\partial s} = -2s$$

Then,

$$\frac{\partial f}{\partial s} = y^2 e^{xy^2} - 4sxye^{xy^2}$$

For $\frac{\partial f}{\partial t}$, the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ can be used from the earlier part.

$$\frac{\partial x}{\partial t} = -2t, \quad \frac{\partial y}{\partial t} = 1$$

Then,

$$\frac{\partial f}{\partial s} = -2ty^2 e^{xy^2} + 2xye^{xy^2}$$

(c) $f(x, y, z) = x^2 + 3y^2 + 2z^2$, $x = t + s$, $y = t^2 + s^2$, $z = t - s$.

Solution. Solution. Using similar procedure as in the previous question (but use formula for function of 3 variables), we obtain

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial s} = 2x + 12ys - 4z,$$

$$\frac{\partial f}{\partial t} = 2x + 12yt + 4z.$$

3. Consider the function $u(x, y, z) = \dfrac{x^4}{y} + \dfrac{y^2}{z^3}$ where $x = rse^t + r$, $y = ste^r + s$ and $z = rte^s + t$. Find $\dfrac{\partial f}{\partial r}$, $\dfrac{\partial f}{\partial s}$ and $\dfrac{\partial f}{\partial t}$.

Solution. For this question, we shall utilize the following formula:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r},$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s},$$

and

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}.$$

Upon computation, we will have

$$\frac{\partial u}{\partial r} = \left(\frac{4x^3}{y}\right)(se^t + 1) + \left(-\frac{1}{y^2} + \frac{2y}{z^3}\right)(ste^r) + \left(-\frac{-3y^2}{z^4}\right)(te^s),$$

$$\frac{\partial u}{\partial s} = \left(\frac{4x^3}{y}\right)(re^t) + \left(-\frac{1}{y^2} + \frac{2y}{z^3}\right)(te^r + 1) + \left(-\frac{-3y^2}{z^4}\right)(rte^s),$$

and

$$\frac{\partial u}{\partial t} = \left(\frac{4x^3}{y}\right)(rse^t) + \left(-\frac{1}{y^2} + \frac{2y}{z^3}\right)(se^r) + \left(-\frac{-3y^2}{z^4}\right)(re^s + 1).$$

4. Find the stated derivatives for the functions given.
For question (a) and (b), we shall use the Implicit Differentiation formula given by:

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

for $F(x, y, z) = 0$.
Meanwhile, for (c) and (d), the following formula is used

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{F_x}{F_z}$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{F_y}{F_z} \tag{1}$$

(a) Find $\frac{dy}{dx}$ if $\sqrt{x+3y} = cos(x^2+2)$

Solution.

$$\frac{dy}{dx} = -\frac{\frac{1}{2}(x+3y)^{-1/2} + \sin(x^2+2).2x}{\frac{3}{2}(x+3y)^{-1/2}}$$

(b) Find $\frac{dy}{dx}$ if $\frac{1}{x^2+1} = tan(x) + xy^2$

Solution.

$$\frac{dy}{dx} = -\frac{2x(x^2+1)^{-2} - \sec^2(x) + y^2}{2yx}$$

(c) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $yz(2y+1) = \sin(xy) - xy^2$.

Solution.

$$\frac{\partial z}{\partial x} = -\frac{-\cos(xy) + y^2}{2y^2 + y}$$

and

$$\frac{\partial z}{\partial y} = -\frac{4yz + z - \cos(xy + 2yx)}{2y^2 + y}$$

(d) Find $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$ if $\ln(2s+1) + \cos^2(2s+tw) = \frac{s}{t}$.

Solution.

$$\frac{\partial w}{\partial s} = -\frac{\frac{2}{2s+1} + 4\cos(2s+tw) - \frac{1}{t}}{2w\cos(2s+tw)}$$

and

$$\frac{\partial 2}{\partial t} = -\frac{2w\cos(2s+tw) + \frac{s}{t^2}}{2w\cos(2s+tw)}$$

5. Find the directional derivative of the function at the point given in the direction of the vector $\mathbf{v}$.

The procedures for Question 5 involves the following steps.
1. We compute the gradient vector, ∇f.
2. find the unit vextor $\mathbf{u}$ from the given $\mathbf{v}$.
3. Compute $D_u f(x_0, y_0) = \nabla f \cdot \mathbf{u}$.

(a) $f(x,y) = 1 + 2x\sqrt{y}$, $(5,6)$, $\mathbf{v} = \langle 9, 16 \rangle$

Solution.

$$\nabla f = \langle f_x, f_y \rangle = \langle 2\sqrt{y}, xy^{-1/2} \rangle$$

Then from $\mathbf{v} = \langle 9, 16 \rangle$, we have

$$\mathbf{u} = \frac{\langle 9, 16 \rangle}{\sqrt{9^2 + 16^2}} = \langle \frac{9}{\sqrt{337}}, \frac{16}{\sqrt{337}} \rangle$$

Thus,

$$
\begin{aligned}
D_u f(x,y) &= \langle 2\sqrt{y}, xy^{-1/2}\rangle \cdot \langle \frac{9}{\sqrt{337}}, \frac{16}{\sqrt{337}}\rangle \\
D_u f(5,6) &= \langle 2\sqrt{6}, (5)(6)^{-1/2}\rangle \cdot \langle \frac{9}{\sqrt{337}}, \frac{16}{\sqrt{337}}\rangle \\
&\approx 1.9099
\end{aligned}
$$

(b) $g(r,s) = \arctan(rs)$, $(1,2)$, $\mathbf{v} = 5\mathbf{i} + 10\mathbf{j}$

Solution.

$$\nabla f = \langle \frac{s}{1+(rs)^2}, \frac{r}{1+(rs)^2}\rangle$$

Then from $\mathbf{v} = \langle 5, 10\rangle$, we have

$$\mathbf{u} = \frac{\langle 5, 10\rangle}{\sqrt{5^2+10^2}} = \langle \frac{5}{\sqrt{125}}, \frac{10}{\sqrt{125}}\rangle$$

Thus,

$$
\begin{aligned}
D_u f(r,s) &= \langle \frac{s}{1+(rs)^2}, \frac{r}{1+(rs)^2}\rangle \cdot \langle \frac{5}{\sqrt{125}}, \frac{10}{\sqrt{125}}\rangle \\
D_u f(1,2) &= \langle \frac{2}{1+(1\times 2)^2}, \frac{1}{1+(1\times 2)^2}\rangle \cdot \langle \frac{5}{\sqrt{125}}, \frac{10}{\sqrt{125}}\rangle \\
&\approx 0.7562
\end{aligned}
$$

(c) $f(x,y,z) = \sqrt{xyz}$, $(3,2,6)$, $\mathbf{v} = \langle -1, -2, 2\rangle$

Solution.

$$\nabla f = \langle f_x, f_y, f_z\rangle = \langle yz(xyz)^{-1/2}, xz(xyz)^{-1/2}, xy(xyz)^{-1/2}\rangle$$

Then from $\mathbf{v} = \langle -1, -2, 2\rangle$, we have

$$\mathbf{u} = \frac{\langle -1, -2, 2\rangle}{\sqrt{(-1)^2+(-2)^2+2^2}} = \langle \frac{-1}{\sqrt{9}}, \frac{-2}{\sqrt{9}}, \frac{2}{\sqrt{9}}\rangle$$

Thus,

$$
\begin{aligned}
D_u f(x,y) &= \langle yz(xyz)^{-1/2}, xz(xyz)^{-1/2}, xy(xyz)^{-1/2}\rangle \cdot \langle \frac{-1}{3}, \frac{-2}{3}, \frac{2}{3}\rangle \\
D_u f(3,2,6) &= \langle (2)(6)((3)(2)(6))^{-1/2}, (3)(6)((3)(2)(6))^{-1/2}, (3)(2)((3)(2)(6))^{-1/2}\rangle \cdot \langle \frac{-1}{3}, \frac{-2}{3}, \frac{2}{3}\rangle \\
&= \langle 2, 3, 1\rangle \langle \frac{-1}{3}, \frac{-2}{3}, \frac{2}{3}\rangle \\
&= \frac{-6}{3} = -2
\end{aligned}
$$

6. From functions in Question 5, find the maximum rate of change for all Subquestions (a), (b) and (c) *(Note: The direction $\boldsymbol{v}$ is not used in this question).*

Solution.
We will just show for question 5(a). The rest can be done using similar steps. We previously obtain that

$$\nabla f(x,y) = \langle 2\sqrt{y}, xy^{-1/2}\rangle\rangle$$

and at $(5,6)$,

$$\nabla f(5,6) = \langle 2\sqrt{6}, (5)(6)^{-1/2}\rangle$$

The maximum rate is given as $|\nabla f|$,

$$\begin{aligned}
|\nabla f(5,6)| &= |\langle 2\sqrt{6}, (5)(6)^{-1/2}\rangle| \\
&= \sqrt{(2\sqrt{6})^2 + \left(\frac{5}{\sqrt{6}}\right)^2} \\
&\approx 5.3072
\end{aligned}$$

7. Show that the equation of the tangent plane to the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ at the point (x_0, y_0, z_0) can be written as

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = 1.$$

Solution. From the given function, we have

$$f_x = \frac{2x}{a^2}, \quad f_y = \frac{2y}{b^2}, \quad f_z = \frac{2z}{c^2}$$

At (x_0, y_0, z_0),

$$f_x(x_0,y_0,z_0) = \frac{2x_0}{a^2}, \quad f_y(x_0,y_0,z_0) = \frac{2y_0}{b^2}, \quad f_z(x_0,y_0,z_0) = \frac{2z_0}{c^2}$$

Substituting the partial derivatives into the equation of tangent plane,

$$\begin{aligned}
&\frac{2x_0}{a^2}(x - x_0) + \frac{2y_0}{b^2}(y - y_0) + \frac{2z_0}{c^2}(z - z_0) = 0 \\
&\frac{2x_0}{a^2}x + \frac{2y_0}{b^2}y + \frac{2z_0}{c^2}z - \left(\frac{2x_0^2}{a^2} + \frac{2y_0^2}{b^2} + \frac{2z_0^2}{c^2}\right) = 0 \\
&\frac{2x_0}{a^2}x + \frac{2y_0}{b^2}y + \frac{2z_0}{c^2}z - 2 = 0
\end{aligned}$$

Therefore,

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = 1$$